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XIV Encuentro Andaluz de Matemática Discreta

Almería 22, 23 y 24 de enero de 2025

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SAL
Supercomputación - Algoritmos

Índice

El XIV Encuentro Andaluz de Matemática Discreta se celebró en Almería los días 22,23 y 24 de enero de 2025, organizado por el Departamento de Matemáticas de la Universidad de Almería, el grupo de Investigación Supercomputación-Algoritmos y el Centro de Desarrollo y Transferencia de Investigación Matemática a la Empresa (CDTIME). Este Encuentro sigue siendo un espacio para poner en contacto a diferentes grupos de investigación andaluces, españoles y del resto del mundo que trabajan en el ámbito de la Matemática Discreta.

Continuamos así con la tradición de estos Encuentros, que comenzaron en el año 1999 en La Rábida (Huelva) y se han celebrado en Los Molares (Sevilla, 2001), San José (Almería, 2003), Utrera (Sevilla, 2005), La Línea de la Concepción (Cádiz, 2007), Galaroza (Huelva, 2009), Carmona (Sevilla, 2011), Sevilla (2013), Almería (2015), La Línea de la Concepción (Cádiz, 2017), Sevilla (2020), Jerez de la Frontera (Cádiz, 2022) y Cádiz (2023).

Queremos agradecer al Departamento de Matemáticas de la Universidad de Almería y al Centro de Desarrollo y Transferencia de Investigación Matemática a la Empresa (CDTIME) su ayuda y patrocinio.

Nuestro agradecimiento también a los participantes, cuya contribución ha hecho de este evento todo un éxito.

El Comité Organizador

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Computación cuántica aplicada a problemas de optimización en grafos

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La computación cuántica es un nuevo paradigma de procesamiento de la información que ofrece ventajas potenciales en campos tan diversos como la criptografía, la simulación de procesos físicos o la inteligencia artificial. Una de las áreas en las que la investigación de las aplicaciones de la computación cuántica es actualmente más intensa es el de la optimización y, más en concreto, la optimización combinatoria.

En esta charla, se ofrecerá una breve introducción a la optimización cuántica, con especial énfasis en los métodos que pueden ser utilizados en problemas provenientes del campo de la matemática discreta. Se tratarán métodos como el Quantum Approximate Optimization Algorithm (QAOA) o el quantum annealing y se ejemplificará su aplicación mediante problemas de optimización sobre grafos como, por ejemplo, el problema del corte máximo.

Visibilidad mutua con restricción de distancias

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El concepto de visibilidad mutua en grafos, que ha sido introducido recientemente, aborda un problema clave en la Teoría de Grafos: cómo identificar el mayor subconjunto posible de vértices en el que cualquier par de ellos esté conectado mediante un camino más corto, que no incluya internamente a ningún vértice del propio subconjunto. Este concepto surgió originalmente a raíz de retos en Informática relacionados con la navegación de robots, donde el objetivo es garantizar que los robots puedan comunicarse sin obstáculos mientras se desplazan.

El problema de visibilidad mutua consiste en encontrar el conjunto más grande de vértices mutuamente visibles en un grafo, y el número de visibilidad mutua de un grafo es simplemente el tamaño de ese subconjunto. Este tema ha atraído mucha atención en la investigación, generando vínculos con problemas combinatorios clásicos como el problema de Zarankiewicz o problemas "de tipo Turán".

En esta charla, nos centramos en las limitaciones prácticas que existen en la visibilidad de redes y extendemos el concepto original a la visibilidad mutua a distancia k. Es decir, dos vértices se consideran S-visibles si existe un camino más corto entre ellos, cuya longitud no exceda de k, y excluyendo de los vértices internos del camino a los propios vértices del conjunto S. El número de k-visibilidad mutua denota al tamaño del conjunto más grande con esta propiedad.

Introduciremos los conceptos básicos asociados al estudio de este nuevo parámetro en grafos y presentaremos algunas características y propiedades del mismo incluyendo la complejidad algorítmica, cotas y valores exactos de este parámetro para ciertos tipos de grafos no triviales.

On the orbital structure of MAX and MIN synchronous multi-state networks *

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Abstract. This work introduces a generalization of Boolean networks to multi-state networks defined over a complement-closed set X, which can be either finite or infinite. The focus is on MAX (and MIN) multi-state networks, whose dynamics are determined by global arbitrary X-maxterm (or X-minterm) functions, extending the classical maxterm (or minterm) Boolean functions. We examine the types of periodic orbits that can arise and coexist, particularly in systems where all vertices have self-loops and the graph is undirected. Our findings reveal that these systems can exhibit only fixed points and 2-periodic orbits, mirroring the behavior observed in Boolean networks. However, a key distinction is that fixed points and periodic orbits can coexist, which contrasts with established results for Boolean networks.

1 Introduction

A Boolean network (BN) with n entities, also referred to as a Boolean finite dynamical system (see, for example, [5, 6]), is defined by an update function

$$\mathbf{F} = (F_1, \ldots, F_n) : \mathcal{B}^n \to \mathcal{B}^n,$$

where the state space $\mathcal{B}$ possesses the (algebraic) structure of a Boolean algebra. When $\mathcal{B} \equiv \{0, 1\}$, the system corresponds to the standard binary Boolean network. Standard (binary) BNs have been widely used to model a variety of phenomena, including gene regulatory networks, social interactions, epidemics, and physical processes, among others. However, their simplicity, which is often a strength, can also pose limitations. Many real-world scenarios require models where entities can assume more than just two off-states. Recently, more general Boolean Algebras (with more than two elements 0/1) have been also studied. Thus, in [2, 3], the authors investigate multi-state (or non-binary) BNs. By using an appropriate algebraic representation of these structures based on the Stone Theorem, it becomes possible to analyze such systems by studying independent copies of the corresponding (binary) BN.

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In this context, some studies have aimed at extending results related to BN to more general networks over finite sets, commonly referred to as multi-state (or multivalued) networks (MN) [7, 10], or finite dynamical systems (FDS) [11]. Another common approaches define the values that the entities can take as elements of a finite field ([8]) or semi-latices [9].

All these systems are typically represented by a graph $G = (V, E)$, where the vertices correspond to the entities and the edges capture the influence between them. The state variables associated with the vertices, the local functions F_i that describe the interactions, and the update schedule for the state values (synchronous, block-sequential or sequential) totally determine the system. In this work we assume that the system is homogeneous (the local functions are the restriction of a global one), that the graph is undirected and all self-loop (the updating of a vertex also depends on itself), an that the update schedule is synchronous.

One of the primary objectives when dealing with networks over finite sets is to analyze its dynamics, with a particular focus on identifying the limit configurations (periodic orbits) to which all other states converge. For instance, for homogeneous synchronous BN (both with binary and non-binary states) whose evolution operator is a Boolean maxterm or minterm, it was shown that the only periodic orbits are fixed points and 2-periodic orbits and they cannot coexist [1, 2].

This work introduces a generalization of Boolean networks to multi-state networks defined over a complement-closed set X, which can be either finite or infinite. In particular we deal with MAX (and MIN) multi-state networks on undirected graphs, whose dynamics are determined by global arbitrary X-maxterm (or X-minterm) functions, extending the classical maxterm (or minterm) Boolean functions. Hence, we show that their only periodic orbits are fixed points and 2-periodic orbits, mirroring the behavior observed in Boolean networks. However in this framework fixed points and periodic orbits can coexist, which contrasts with established results for Boolean networks.

2 Results

Let $(X, \leq)$ be a totally ordered set. As usual, we write $a < b$ (or $b > a$) meaning that $a \leq b$ and $a \neq b$. Let $' : X \to X$ be a bijective map satisfying that (i) For every $a, b \in X$, if $a < b$ then $b' < a'$; and (ii) For every $a \in X$, $a'' = a$. Then, we say that the triple $(X, \leq, ')$ is a *complement-closed set*. From this definition, if there exists $c \in X$ such that $c' = c$, this element is unique and is called the *central value* of X.

Let X be a complement-closed set and $G = (V, E)$ a simple, connected, undirected graph with $V = \{1, 2, \ldots, n\}$. The map

$$\mathbf{F} = (F_1, \ldots, F_n) : X^n \to X^n, \;\; \mathbf{F}(x_1, x_2, \ldots, x_i, \ldots, x_n) = (y_1, y_2, \ldots, y_i, \ldots, y_n),$$

where y_i is the updated state of the vertex i by applying the local function F_i over the states of the vertices adjacent to i, is called a *synchronous multi-state network (*SyMN*) over* X.

The following ones are particular examples of SyMN:

- If $X = \{0, 1\}$, $\mathbf{F}$ is a standard synchronous binary BN [1, 5, 6].

- If $X = [0, 1]$, $\mathbf{F}$ is a fuzzy synchronous dynamical system on Zadeh operators [4].
- If $X = \{0, 1, \ldots, q\} \subseteq \mathbb{Z}$, $\mathbf{F}$ is a synchronous FDS [8, 11].

Throughout this work, we will choose the global functions $F : X^n \to X$ as the following natural generalizations of the maxterms and minterms Boolean functions: we define an X-*maxterm* function on n variables MAX as

$$\mathrm{MAX}(x_1, \ldots, x_n) = \max\{z_1, \ldots, z_n\}, \qquad \text{where } z_i = x_i \text{ or } z_i = x_i',$$

and analogously an X-*minterm* function on n variables MIN as

$$\mathrm{MIN}(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} \qquad \text{where } z_i = x_i \text{ or } z_i = x_i'.$$

When F is an X-maxterm, MAX (respectively an X-minterm, MIN), we say that (X, G, MAX) is a MAX-SyMN (resp. MIN-SyMN).

Then we have:

Theorem 2.1. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN. *Then, all the periodic points of the system are fixed points or* 2*-periodic points.*

Given $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ a MAX-SyMN, let us consider the subsets of vertices

$$W = \{i \in V : x_i \text{ appears in direct form in } F\}, \quad W' = \{i \in V : x_i \text{ appears in complemented form in } F\}$$

and

$$W_C' = \{i \in W' : A_G(i) \cap W = \emptyset\},$$

where $A_G(i)$ is the set of vertices adjacent to i. Then:

Theorem 2.2. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN. *Assume that* $W_C' = \emptyset$. *Then, all the periodic orbits of the system are fixed points.*

Theorem 2.3. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN *with no central value. Assume that* $W_C' \neq \emptyset$. *Then, all the periodic orbits of the system are* 2*-periodic orbits.*

Theorem 2.4. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN *and assume that there exists a central value* $c \in X$. *Suppose that* $W_C' \neq \emptyset$. *Then, the system presents fixed points and* 2*-periodic orbits.*

Corollary 2.5. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN *and assume that* X *has no central value. Then, all the periodic orbits of the system are fixed points if, and only if,* $W_C' = \emptyset$. *On the other hand, all the periodic points of the system are* 2*-periodic orbits if, and only if,* $W_C' \neq \emptyset$. *That is, fixed points and* 2*-periodic orbits cannot coexist.*

Corollary 2.6. *Let* $\mathbf{F} \equiv (X, G, \mathrm{MAX})$ *be a* MAX-SyMN *and assume that there exists a central value* $c \in X$. *Then, the system presents fixed points. Moreover,* $\mathbf{F}$ *presents* 2*-periodic orbits if, and only if,* $W_C' \neq \emptyset$. *That is, fixed points and* 2*-periodic orbits coexist provided that* $W_C' \neq \emptyset$.

3 Conclusions

In this work, we study the orbital structure of homogeneous MAX synchronous multi-state networks over complement-closed sets. We prove that, similar to MAX Boolean networks, these systems only present fixed points and 2-periodic orbits. However, unlike Boolean systems, fixed points and 2-periodic orbits can coexist. All these results can be adapted immediately for MIN multi-state networks over complement-closed sets.

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Digraphs associated with Tortkara algebras*

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Abstract. In this paper, we study the relation between Tortkara algebras and combinatorial structures. We will focus on the properties that have to be satisfied by those combinatorial structures so that they are associated with low-dimensional Tortkara algebras.

1 Introduction

Tortkara algebras are a new class of non-associative algebras introduced by Dzhumadildaev in [1], where it is proved that every Zinbiel algebra with the commutator multiplication gives a Tortkara algebra.

Our main goal is to study the relations between Graph Theory and Tortkara algebras as it was done in [2, 3] with Leibniz and Zinbiel algebras.

2 Preliminaries

We show some preliminary concepts on Tortkara algebras.

Definition 2.1. *A Tortkara algebra $\mathcal{T}$ over a field $\mathbb{K}$ is a vector space with a bilinear product which satisfies*

$$xx = 0,\ \forall x \in \mathcal{T}, \tag{1}$$

$$(xy)(zy) = J(x,y,z)y,\ \textit{where}\ J(x,y,z) = (xy)z + (yz)x + (zx)y,\ \forall x,y,z \in \mathcal{T}. \tag{2}$$

The latter is called the Tortkara identity.

Corollary 2.2. *A Tortkara algebra is anticommutative (i.e. $xy = -yx$, $\forall x, y \in \mathcal{T}$).*

Proposition 2.3. *Over a field of characteristic different from two, the Tortkara identity has the following multi-linear form*

$$(xy)(zw) + (xw)(zy) = J(x,y,z)w + J(x,w,z)y,\ \forall x,y,z,w \in \mathcal{T}. \tag{3}$$

From now on, we use the notation

$$T(x,y,z,w) = (xy)(zw) + (xw)(zy) - J(x,y,z)w - J(x,w,z)y.$$

*Acknowledgments: This work has been partially supported by FQM-326.

Definition 2.4. *Given a basis $\{e_i\}_{i=1}^n$ of an n-dimensional Tortkara algebra $\mathcal{T}$, its structure constants are defined by $e_i e_j = \sum_{h=1}^{n} c_{i,j}^h e_h$, for $1 \leq i, j \leq n$.*

Definition 2.5. *The derived series of a finite-dimensional Tortkara algebra $\mathcal{T}$ is*

$$\mathcal{T}_1 = \mathcal{T},\ \mathcal{T}_2 = \mathcal{T}\mathcal{T},\ \mathcal{T}_3 = \mathcal{T}_2\mathcal{T}_2,\ \ldots,\ \mathcal{T}_k = \mathcal{T}_{k-1}\mathcal{T}_{k-1},\ \ldots$$

$\mathcal{T}$ is $(m-1)$-step solvable if there exists $m \in \mathbb{N}$ such that $\mathcal{T}_m = \{0\}$ and $\mathcal{T}_{m-1} \neq \{0\}$.

Definition 2.6. *The central series of a finite-dimensional Tortkara algebra $\mathcal{T}$ is*

$$\mathcal{T}^1 = \mathcal{T},\ \mathcal{T}^2 = \mathcal{T}\mathcal{T},\ \mathcal{T}^3 = \mathcal{T}^2\mathcal{T},\ \ldots,\ \mathcal{T}^k = \mathcal{T}^{k-1}\mathcal{T},\ \ldots$$

$\mathcal{T}$ is $(m-1)$-step nilpotent if there exists $m \in \mathbb{N}$ such that $\mathcal{T}^m = \{0\}$ and $\mathcal{T}^{m-1} \neq \{0\}$.

3 Associating combinatorial structures with Tortkara algebras

Let $\mathcal{T}$ be a n-dimensional Tortkara algebra with basis $\mathcal{B} = \{e_i\}_{i=1}^n$ and whose structure constants correspond to $e_i e_j = \sum_{h=1}^n c_{i,j}^h e_h$. Then, the pair $(\mathcal{T}, \mathcal{B})$ is associated with a combinatorial structure by the following procedure:

a) For each $e_i \in \mathcal{B}$, we draw a vertex i.

b) Given two vertices i, j with $i < j$ verifying $c_{i,j}^j \neq 0$, we draw a directed edge from vertex i to j with weight $c_{i,j}^j$. Also, if $c_{i,j}^i \neq 0$, we draw a directed edge from vertex j to i with weight $c_{i,j}^i$.

c) Given three vertices $i < j < k$ such that $(c_{i,j}^k, c_{j,k}^i, c_{i,k}^j) \neq (0,0,0)$, we draw a full triangle ijk such that the edges ij, jk and ik have weights $c_{i,j}^k$, $c_{j,k}^i$ and $c_{i,k}^j$, respectively. Moreover:

   c1) We use a discontinuous line (named *ghost edge*) for edges with weight 0.

   c2) If two triangles ijk and ijl with $i < j < k < l$ satisfy $c_{i,j}^k = c_{i,j}^l$, draw only one edge between vertices i and j shared by both triangles.

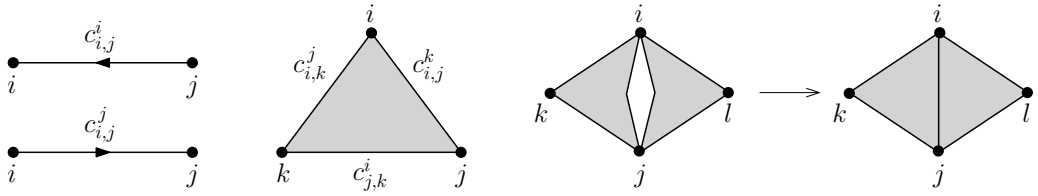

Figure 1: Directed edge, full triangle and two triangles sharing an edge.

4 Digraphs and Tortkara algebras

We study the digraphs associated with Tortkara algebras. Let $\mathcal{T}$ be a Tortkara algebra with basis $\mathcal{B} = \{e_i\}_{i=1}^n$ such that the combinatorial structure G associated with $(\mathcal{T}, \mathcal{B})$ consists of a digraph; that is, there are no triangles in G. The law of $\mathcal{T}$ is given by

$$\begin{gathered} e_i e_j = c_{i,j}^i e_i + c_{i,j}^j e_j = -e_j e_i,\ 1 \leq i < j \leq n, \\ e_k e_k = 0,\ 1 \leq k \leq n. \end{gathered} \tag{4}$$

The only possible digraph G with 1 vertex is formed by an isolated vertex and it is associated with the unique 1-dimensional Tortkara algebra, which is abelian.

Proposition 4.1. *Let G be a digraph of 2 vertices associated to a 2-dimensional Tortkara algebra $\mathcal{T}$ with basis $\mathcal{B} = \{e_i\}_{i=1}^2$. Then, G can be isomorphic to every possible configuration, which are shown in Figure 2, if and only if all edges have non-zero weight.*

Proposition 4.2. *Under the assumptions in Proposition 4.1, it is verified that*

- *Configuration $a)$ is associated with the abelian 2-dimensional Tortkara algebra.*
- *Configurations $b)$ and $c)$ are associated to 2-step solvable, non-nilpotent Tortkara algebras.*

Proposition 4.3. *Let G be a non-connected digraph of 3 vertices associated to a 3-dimensional Tortkara algebra $\mathcal{T}$ with basis $\mathcal{B} = \{e_i\}_{i=1}^3$. Then, G can be isomorphic to every possible configuration, which are shown in Figure 3, if and only if all edges have non-zero weight.*

Proposition 4.4. *Under the assumptions in Proposition 4.3, it is verified that*

- *Configuration $i)$ is associated with the abelian 3-dimensional Tortkara algebra.*
- *Configurations $ii)$ and $iii)$ are associated to 2-step solvable, non-nilpotent Tortkara algebras.*

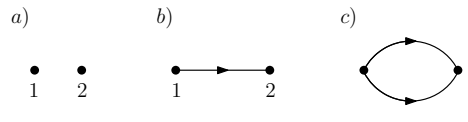

Figure 2: Digraphs with two vertices.

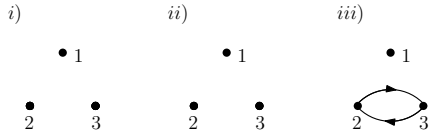

Figure 3: Disconnected digraphs with three vertices.

Proposition 4.5. *Let G be a connected digraph of 3 vertices associated to a 3-dimensional Tortkara algebra $\mathcal{T}$ with basis $\mathcal{B} = \{e_i\}_{i=1}^3$. Then, G cannot be isomorphic to configurations 7), 11) and 12) shown in Figure 4. Any other configuration is associated to a Tortkara algebra if and only if all edges have non-zero weight and, in case of configuration 13), it must also satisfy $c_{1,2}^1 = -\dfrac{c_{1,2}^2 c_{1,3}^1 c_{2,3}^3}{c_{1,3}^3 c_{2,3}^2}$.*

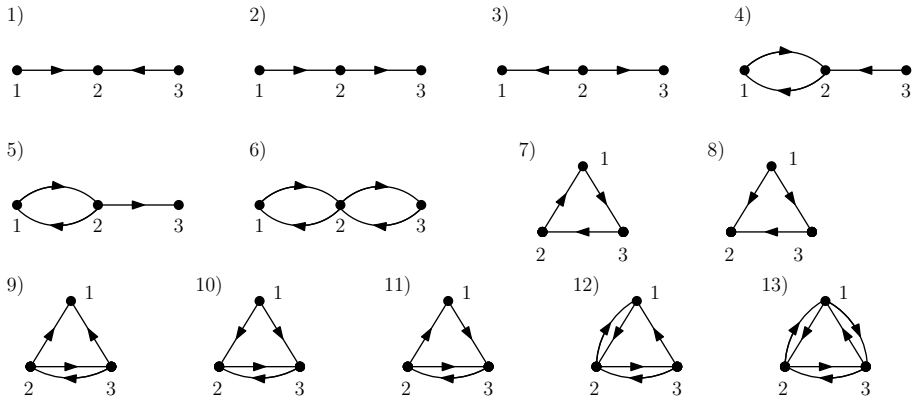

Figure 4: Connected digraphs with three vertices.

Proposition 4.6. *Under the assumptions in Proposition 4.5, all valid configurations are associated to solvable, non-nilpotent Tortkara algebras. More precisely:*

- $1)$ *and* $3)$ *are associated to* 2*-step solvable algebras.*
- $2)$, $4)$, $5)$, $6)$, $8)$, $10)$ *are associated to* 3*-step solvable algebras.*
- $9)$ *is associated to a* 2*-step solvable algebra if* $c^1_{1,2} = -\dfrac{c^1_{1,3}c^3_{2,3}}{c^2_{2,3}}$*. Otherwise, it is* 3*-step solvable.*
- $13)$ *is associated to a* 2*-step solvable algebra if* $c^2_{1,2} = c^3_{1,3}$ *and* $c^2_{2,3} = c^1_{1,3}$*. Otherwise, it is* 3*-step solvable.*

5 Algorithm for the Tortkara identity

We have developed an algorithmic procedure used in the previous section to evaluate the Tortkara identity and find out the forbidden and allowed configurations. We start considering a vector space $\mathcal{T}$ with basis $\mathcal{B}$ and the law expressed in (4). We have implemented our algorithm using the symbolic computation package MAPLE 22 following these steps:

1. Computing the product between two arbitrary basis vectors in $\mathcal{B}$.
2. Evaluating the product between two vectors expressed as a linear combination of vectors from $\mathcal{B}$.
3. Imposing the multi-linear form of the Tortkara identity (Equation 3) to every possible combination of elements of $\mathcal{B}$ and solving the corresponding system of equations.

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Localización de una recta mediana con pesos generales *

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Resumen. Dado un conjunto P de n puntos en el plano, el problema de la recta mediana consiste en encontrar una recta r que minimice la suma de distancias ponderadas de r a los puntos de P. En este trabajo estudiamos el problema que resulta cuando los pesos, asignados a los puntos de P, pueden ser positivos o negativos y la distancia considerada es la distancia euclídea. Asimismo, describimos un algoritmo que encuentra una solución al problema en tiempo polinomial.

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1. Introducción

Dado un conjunto $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ de n puntos de demanda, que supondremos en posición general, y un conjunto de pesos $W = \{w_1, \ldots, w_n\}$, asociados a los elementos de P, el problema de la recta mediana consiste en encontrar una recta r^* que haga mínima la función $f(r) = \sum_{i=1}^{n} w_i d(p_i, r)$, donde $d(\cdot, \cdot)$ denota la distancia euclídea punto-recta.

Cuando los pesos son no negativos, esto es, $w_i \geq 0$, $i = 1, \ldots, n$, el problema ha sido ampliamente estudiado en la literatura. Así, en [3, 4, 6, 7] se demuestra el siguiente resultado:

Propiedad 1.1. *Si r^* es una recta mediana de un conjunto de puntos y denotamos por $H^+_{r^*}$ y $H^-_{r^*}$ a los dos semiplanos abiertos, definidos por r^*, entonces se cumplen:*

$$i) \sum_{p_i \in P \cap H^+_{r^*}} w_i \leq \frac{\sum_{i=1}^{n} w_i}{2} \qquad ii) \sum_{p_i \in P \cap H^-_{r^*}} w_i \leq \frac{\sum_{i=1}^{n} w_i}{2}.$$

También en [3, 4, 6, 7] se demuestra una segunda propiedad de la recta mediana que es muy importante para el desarrollo de los primeros algoritmos diseñados para resolver el problema y que enunciamos a continuación:

Propiedad 1.2. *Existe, al menos, una recta solución del problema mediana que pasa por dos puntos de P.*

Haciendo uso de esta propiedad, Morris y Norback [6], proponen un algoritmo que resuelve el problema en tiempo $O(n^3)$.

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Posteriormente, en [5], Megiddo y Tamir proponen un algoritmo de complejidad $O(n^2 \log n)$ y, finalmente, en [8], Yamamoto et al. proponen un algoritmo que resuelve el problema en tiempo $O(n^2)$. Una amplia descripción de la bibliografía relacionada con este problema puede encontrarse en [1].

Cuando no todos los elementos de W son positivos resulta un problema sustancialmente distinto. En este caso, como se demuestra en [2], existirá al menos una solución al problema que necesariamente pasa por un punto de P pero no tenemos garantizado que exista al menos una solución que pase por dos puntos distintos de P. Esto es, la propiedad 1.2 no se cumple.

A pesar de esto, hasta donde conocemos, esta versión del problema casi no ha sido estudiada en la bibliografía y no existe ninguna referencia en la que se describa un algoritmo que permita resolverla de forma exacta. Únicamente, en [2], se estudian algunas de sus propiedades y se resuelve el problema mediante un algoritmo de optimización metaheurístico basado en enjambre de partículas (particle swarm optimization algorithm).

En este trabajo estudiamos el problema de la recta mediana cuando se considera la distancia euclídea y no todos los pesos en W tienen el mismo signo. Profundizamos en las propiedades del problema y describimos un algoritmo exacto, basado en el cálculo del nivel mediano en un arreglo de rectas en el plano, que permite resolverlo en tiempo polinomial.

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Normal Ordering of $(UD)^n$ when $DU = qUD + D + 1$ (extended abstract)*

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Abstract. We describe in several ways the Normal Ordering of $(UD)^n$ when $DU = qUD + D + 1$ and commute with q. This generalizes a result of Mansour and Schork.

1 Introduction

The original *Normal Ordering Problem* is about finding the unique expansion as linear combination of ordered monomials $U^i D^j$ (all U before all D) of a polynomial $P(U, D)$, where U and D satisfy the commutation relation $DU - UD = 1$, and no other independent relation. Such pairs (U, D) appear in analysis as U =multiplication by a variable x with D = derivation with respect to x), on a space (big enough) of smooth functions of one variable. They also appear in quantum physics as a pair U = creation operator $\hat{b}^\dagger$ with D = deletion operator $\hat{b}$. The algebra generated by two such elements U and D is the *Weyl algebra*.

The Normal Ordering Problem has been extensively studied, see the survey [1]. A special attention has been paid to the normal ordering of $(UD)^n$, whose expansion was found (already in the early XIX$^{\text{th}}$ century [2], see also [3] for a modern reference) to be

$$(UD)^n = \sum_{k=0}^{n} S(n,k) U^k D^k \tag{1}$$

where the coefficients $S(n, k)$ are the well-known Stirling numbers of the second kind. Since these numbers are also the coefficients in the expansion in falling factorials $(x)_k = x \cdot (x-1) \cdots (x-k+1)$ of the monomials:

$$x^n = \sum_{k=0}^{n} S(n,k) \, (x)_k$$

we see that Relation (1) can be summarized in a more compact way as:

$$U^k D^k = (UD)_k \, . \tag{2}$$

Variants of the normal ordering problem have been considered as well, with modified commutation relations. Changing the relation $DU - UD = 1$ into $DU - qUD = 1$, where q is an element commuting with both of D and U, gives rise to a q-analogue of the classical Normal

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Ordering Problem. Such a pair of operators appears when taking for U the multiplication by a variable x, and for D the q-derivative defined by:

$$f(x) \mapsto \frac{f(qx) - f(x)}{qx - q},$$

where q is a scalar or a variable independent from x.

In [4], the Normal Ordering Problem is widely generalized, where the relation $DU - UD = 1$ is replaced with

$$DU - UD = \alpha D + \beta U + \gamma,$$

or with

$$DU - qUD = \alpha D + \beta U + \gamma. \tag{3}$$

A particular case of interest is with the commutation relation

$$DU - UD = D + 1. \tag{4}$$

An example of a pair (U, D) fulfilling (4) is given by $U =$ multiplication by x and $D =$forward divided difference

$$f(x) \mapsto f(x+1) - f(x)$$

Mansour and Schork provided in [5] the following solution to the Normal Ordering Problem for $(UD)^n$ in this context:

$$(UD)^n = \sum_{j,k} S(n,k)c(k,j)U^j D^k \tag{5}$$

where the $S(n,k)$ are the Stirling number of the second kind introduced earlier, and the $c(n,k)$ are the signless Stirling numbers of the first kind[1] , that can be defined as the coefficients in the expansion of the rising factorials $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ in monomials:

$$(x)^{(n)} = \sum_k c(n,k)x^k.$$

Mansour and Schork ask finally [5, Section 5, Remark 7] for a q-analogue of this normal ordering. We answer here that question and explore some related issues. A detailed version of this work, including proofs, is in preparation [6].

2 Normal ordering of $(UD)^n$ when $DU = qUD + D + 1$

We start with the observation that, similarly to the restatement of (1) into (2), the expansion (5) can be restated compactly as

$$(UD)_n = (U)^{(n)} D^n$$

Proving (5) under this compact form provides enough insight to derive a q-analogue, that we present below.

[1]Mansour and Schork actually use the Stirling numbers of the first kind $s(k,j)$, but these are related to their signless alter ego by the simple relation $s(k,j) = (-1)^{k-j}c(k,j)$.

In this aim, let us recall the definition of the classical q-integers:

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$$

Let us also introduce the following analogues of the falling and raising factorials:

$$(z)_{n,q} = z(z - [1]_q)(z - [2]_q) \cdots (z - [n-1]_q)$$

and

$$(z)^{((n,q))} = z(qz + [1]_q)(q^2 z + [2]_q) \cdots (q^{n-1} z + [n-1]_q).$$

Theorem 2.1. *Let U, D and q be elements of an algebra with unit, such that q commutes with D and U, and*

$$DU = qUD + D + 1. \tag{6}$$

Then, for all $n \geq 0$,

$$(UD)_{n,q} = (U)^{((n,q))} D^n.$$

This theorem provides an expansion analogue to (5) that onvolves q-analogues of the Stirling coefccients of both kinds.

Note that the commutation relation (6) is actually quite general: if U and D fulfill instead

$$DU = qUD + \alpha D + \gamma,$$

with $\alpha \neq 0$, then $U' = \frac{1}{\alpha} U$ and $D' = \frac{\alpha}{\gamma} D$ fulfill (6).

The interest of introducing the parameter q in the commutation relation is to obtain further results by specialization. Of course, for $q = 1$ we get back the expansion (5). The specialization $q = -1$ corresponds to the anticommutation relation: $UD + DU = D + 1$. We derive the following result for this setting:

Corollary 2.2. *Let U and D such that $DU + UD = D + 1$. Then,*

$$(UD)^n = \sum_{(j,k)} (-1)^{\binom{k+1}{2}+j} \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \binom{\lfloor k/2 \rfloor}{k-j} U^j D^k.$$

3 Combinatorial interpretation

In the classical setting of the Weyl algebra, Anna Varvak [7] has provided a combinatorial interpretation for the coefficients in the Normal Ordering of $(UD)^n$ in terms of *non-attacking rook placements*:

$$(UD)^n = \sum_R U^{r(R)} D^{c(R)}$$

where the sum is over all ways R of placing "rooks" on the triangular array

$$T_n = \left\{ (i,j) \in \mathbb{Z}^2 \,|\, i \geq 1 \text{ and } j \geq 1 \text{ and } \; i + j \leq n \right\}$$

in such a way that no column and no row contains two or more rooks, and $r(R)$ (resp. $c(R)$) is the number of rook-free rows (resp. columns).

Using the general techniques sketched in [3], we obtain a similar (although more intricate) description.

Theorem 3.1. *When* $DU = qUD + D + 1$,

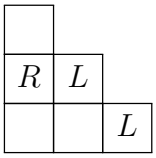
$$(UD)^n = \sum_R q^{a(R)} U^{\ell_1(R)} D^{\ell(R)}$$

where the sum if over all ways of placing "rooks" and "lances[2]*" on* T_n *such that:*

- *each column contains exactly one rook or lance.*
- *there is no rook in the first row (*$j = 1$*).*
- *in any row, if there is a rook, then there is only one, and all lances in this row at to its right*[3]

and with $\ell_1(R)$ *the number of lances in the first row;* $\ell(R)$ *the total number of lances; and* $a(R)$ *the number of empty positions that are above some lance or rook, but not to the left of any rook in the same row.*

For instance, the filling



(where R is for rook and L for lance) gives a term qUD^2 in the normal ordering of $(UD)^3$.

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Una generalización del intercambio de clave en grupo de Burmester-Desmedt basado en la acción de un grupo finito no abeliano

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Resumen. La futura llegada a gran escala de los ordenadores cuánticos tendrá como consecuencia que la infraestructura criptográfica de clave pública existente pueda no ser segura. Este hecho implica que la privacidad de multitud de apliaciones móviles que involucran grupos dinámicos como aquellas de mensajería en grupo o "pay-per-view" podría estar comprometida. En el presente trabajo proponemos una generalización del conocido intercambio de clave en grupo de Burmester and Desmedt al caso no conmutativo a través de la acción de un grupo finito no abeliano, y probamos que el protocolo presentado es seguro en el modelo de Katz y Yung.

1. Introducción

El objetivo de cualquier protocolo de intercambio de clave en grupo (GKE) es la de permitir una comunicación segura a través de una red no confiable a través de la generación y la distribución de claves compartidas por dos o más participantes. Debemos distinguir entre dos tipos de protocolos de intercambio de clave en grupo: los protocolos de transporte de clave (GKT), en los que un participante distinguido genera una clave que es transferida al resto de participantes, y los protocolos de acuerdo de clave (GKA) en los que la clave común se genera a partir de las aportaciones de todos los participantes.

El primer protocolo GKA basado en el acuerdo de clave de Diffie-Hellman fue propuesto por Ingemarsson, Tang en [1]. Aunque en los años subsiguientes una amplia variedad protocolos de intercambio de clave en grupo vieron la luz, el propuesto por Burmester y Desmedt (BD) en [2] ha destacado en el tiempo por su simplicidad y por el pequeño número de rondas necesarias para su ejecución. Si bien es cierto que la publicación de una prueba rigurosa de la seguridad del mismo se dilató hasta la aparición de los trabajos [3] y [4].

Ante los retos de seguridad que supondrá la llegada a gran escala de los ordenadores cuánticos, la teoría de grupos, y en particular la teoría de grupos no abelianos, constituye un vasto área que proporciona una rica y amplia variedad de poblemas alineados con la propuesta de candidatos para la criptografía post-quántica (PQC) (véase p.ej. [5]). Motivados por este hecho, e inspirados por los trabajos [6] y [7], en los que se proponen protocolos de intercambio de clave basados en la acción de un semigrupo , proponemos una generalización del protocolo de

acuerdo de clave BD mediante el uso de la acción de grupos finitos no abelianos, el cual satisface la propiedad de corrección y, en términos de seguridad, la propiedad de confidencialidad directa en el modelo de Katz y Yung [4].

2. Resultados

A continuación proponemos el siguiente protocolo de intercambio de clave en grupo que generaliza el conocido protocolo de Burmester-Desmedt al caso no conmutativo.

Sean $(G, \cdot)$ y $(H, \odot)$ dos grupos finitos. El centro elige entonces un elemento $g \in G$ y lo publica.

Protocolo 2.1. *Sean $U_1, U_2, ..., U_n$ un conjunto de participantes que desean generar una clave privada en grupo.*
[***Ronda 1***] *Los participantes U_i and $U_{i+1mod(n)}$, $i = 1, ..., n$ realizan un intercambio de clave y generan una clave privada $c_i \in H$.*
[***Ronda 2***] *Cada participante U_i, $i = 1, ..., n$ selecciona un elemento $h_i \in H$ y envía $v_i = \phi(h_i, g)$ a $U_{i+1mod(n)}$ y $U_{i-1mod(n)}$.*
[***Ronda 3***] *Cada participante U_i, $i = 1, ..., n$ envía $w_i = \phi(c_{i-1} \odot h_i, v_{i-1mod(n)})$ a $U_{i-1mod(n)}$.*
[***Ronda 4***] *Cada participante U_i, $i = 1, ..., n$ calcula $Y_i = \phi(c_i^{-1}, w_{i+1}) = \phi(h_{i+1mod(n)} \odot h_i, g)$, y $X_i = \phi(h_i, v_{i-1}) = \phi(h_i \odot h_{i-1mod(n)}, g)$.*
[***Ronda 5***] *Cada participante U_i, $i = 1, ..., n$ envía $Z_i = X_i^{-1} \cdot Y_i$.*
[***Cálculo de la clave***] *Cada participante U_i, $i = 1, ..., n$ calcula la clave privada común*

$$K_i = A^i_{\sigma^{i-1}(1)} \cdot A^i_{\sigma^{i-1}(2)} \cdot A^i_{\sigma^{i-1}(3)} \cdots A^i_{\sigma^{i-1}(n)}, \tag{1}$$

donde

$$\begin{aligned}
&A^i_1 \equiv X_i = \phi(h_i \odot h_{i-1mod(n)}, g),\\
&A^i_2 \equiv A^i_1 \cdot Z_i = \phi(h_i \odot h_{i-1mod(n)}, g) \cdot \phi(h_i \odot h_{i-1mod(n)}, g)^{-1} \cdot Y_i,\\
&A^i_3 \equiv A^i_2 \cdot Z_{i+1mod(n)} = \phi(h_{i+1mod(n)} \odot h_i, g) \cdot \phi(h_{i+1mod(n)} \odot h_i, g)^{-1} \cdot Y_{i+1mod(n)},\\
&.....,\\
&A^i_n \equiv A^i_{n-1} \cdot Z_{i+n-2mod(n)},\\
&= \phi(h_{i+n-2mod(n)} \odot h_{i+n-3mod(n)}) \cdot \phi(h_{i+n-2mod(n)} \odot h_{i+n-3mod(n)}, g)^{-1} \cdot Y_{i+n-2mod(n)},
\end{aligned}$$

y $\sigma \in S^n$ es la permutación $\sigma = (1, n, n-1, \ldots, 2)$.

Nótese que en el caso de la acción de un grupo conmutativo, podemos prescindir de las Rondas 1 y 3 del Protocolo 2.1, así como de la permutación σ. Como consecuencia, en un ambiente conmutativo, basta con que $(H, \odot)$ tenga una estructura de semigrupo.

Ejemplo 2.2 (Caso H abeliano)**.** *Sea $G = \mathbb{Z}_p$, g un elemento de G, donde $q = ord(g)$, y $H = \mathbb{Z}_q$. Podemos definir entonces la acción de semigrupo $\phi : H \times G \to G$, donde $\phi(h_i, g) = g^{h_i}$. Si ejecutamos el Protocolo 2.1 obtenemos que $v_i = g^{h_i}$ y $Z_i = g^{h_{i+1} \odot h_i} \cdot (g^{h_i \odot h_{i-1}})^{-1}$, por lo que la clave privada generada en grupo resulta ser*

$$sk = g^{h_1 \odot h_n} \cdot g^{h_2 \odot h_1} \cdots g^{h_n \odot h_{n-1}},$$

obteniendo así el protocolo original de Burmester-Desmedt.

Se dice que un protocolo satisface la condición de corrección ("correctness") si una vez ejecutado todos los participantes obtienen la misma clave compartida.

Proposición 2.3. *[8, Proposition 5.1.] El protocolo 2.1 satisface la condición de correción.*

En lo relativo a la seguridad, afirmamos que el suceso $Succ$ acontece si el adversario $\mathcal{A}$ invoca al oráculo $Test$, y $\mathcal{A}$ acierta el valor del bit b empleado por el oráculo $Test$ en respuesta a su demanda. Así la ventaja de $\mathcal{A}$ en el ataque a un protocolo P se define como

$$Adv_{\mathcal{A},P}(k) \equiv \left|2 \cdot Pr\left[Succ\right] - 1\right|. \tag{2}$$

Decimos entonces que P es un protocolo de intercambio de clave en grupo seguro si es seguro con respecto al ataque de un adversario pasivo, esto es, para cualquier PPT adversario pasivo $\mathcal{A}$ se verifica que la probabilidad $Adv_{\mathcal{A},P}(k)$ es despreciable. Con el fin de realizar un análisis de seguridad, se define $Adv_P^{KE-fs}(t, q_{ex})$ como la mayor ventaja de cualquier adversario pasivo que ataque el protocolo P, en un tiempo t, y que realice q_{ex} invocaciones al oráculo $Execute$. Se dice entonces que el protocolo P posee la condición de confidencialidad directa ("forward secrecy") si la probabilidad $Adv_P^{KE-fs}(t, q_{ex})$ es despreciable.

La seguridad del Protocolo 2.1 está basada en el problema decisional de Diffie-Hellman para acciones de grupo (DDH-GA). Dados $(G, \cdot)$, $(H, \odot)$ dos grupos finitos, $\phi : H \times G \to G$ una acción de un grupo finito, y un elemento $g \in G$, el problema decisional de Diffie-Hellman para la acción phi en distinguir con una ventaja no despreciable entre las distribuaciones

$$(\phi(x,g), \phi(y,g), \phi(y \odot x, g), \phi(x \odot y, g)) \text{ and } (\phi(x,g), \phi(y,g), \phi(z,g), \phi(r,g)), \tag{3}$$

donde $z, r \in H \setminus \{y \odot xH_g, x \odot yH_g\}$ son seleccionados aleatoriamente.
De manera más formal, se define $Adv_G^{ddh-ga}(t)$ como el valor máximo valor, considerando todos los algoritmos distintivos B ejecutados en un tiempo a lo sumo t, de la diferencia:

$$\begin{aligned}&\left|Pr\left[x, y \leftarrow H : B(\phi(x,g), \phi(y,g), \phi(y \odot x, g), \phi(x \odot y, g) = 1\right] - \right.\\ &\left. Pr\left[x, y \leftarrow H; z, r \leftarrow H \setminus \{y \odot xH_g, x \odot yH_g\} : B(\phi(x,g), \phi(y,g), \phi(z,g), \phi(r,g)) = 1\right]\right|.\end{aligned}$$

Se dice entonces que la acción finita $\phi : H \times G \to G$ satisface la hipótesis $DDH - GA$ si la probabilidad $Adv_G^{ddh-ga}(t)$ es "pequeña" para valores "razonables" de t.

Teorema 2.4. *[8, Theorem 5.2.] Dada una acción finita $\phi : H \times G \to G$ que satisfaga el supuesto $DDH - GA$ se verifica que el Protocolo 2.1 es un protocolo de acuerdo de cable seguro que satisface la condición de confidencialidad directa, esto es,*

$$Adv_P^{KE-fs}(t, q_{ex}) \leq 4 \cdot Adv_G^{ddh-ga}(t') + \frac{2 \cdot q_{ex} \cdot |H_g|}{|H|},$$

donde $t' = t + \mathcal{O}(|\mathcal{P}| \cdot q_{ex} \cdot t_\phi)$, $\mathcal{P}$ denota el tamaño del conjunto de potenciales participantes y t_ϕ denota en tiempo necesario para computar $\phi(h, g)$.

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Sobre el espectro de matrices de Jacobi casi-Toeplitz*

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Resumen. Presentamos una metodología para determinar las características espectrales de matrices de Jacobi y la aplicamos para el caso particular en el que las diagonales de tales matrices están formadas por sucesiones periódicas. Aquí, el término *matriz de Jacobi* hace referencia a una matriz tridiagonal simétrica, cuya diagonal secundaria tiene entradas negativas, es decir una Z-matriz tridiagonal e irreducible.

Nuestro análisis espectral para matrices de Jacobi está basado en el estudio de los problemas de contorno de tipo Sturm-Liouville asociados a ecuaciones en diferencias lineales de segundo orden. En particular, incorporamos nuestros resultados sobre ecuaciones en diferencias de segundo orden y con coeficientes periódicos, lo que permite describir las características espectrales de las denominadas *matrices de Jacobi-casi Toeplitz*. A modo de ejemplo, presentamos resultados explícitos para períodos 2 a lo sumo, unificando así muchos resultados dispersos en la literatura matemática.

1. Introducción

Para cada $n \in \mathbb{N}^*$, los vectores en $\mathbb{R}^{n+1}$ serán denotados por $\mathsf{v} = (v_0, \ldots, v_n)$, (siempre con el subíndice comenzando en $k = 0$). Con esta notación, $\mathsf{v} > 0$ significa que $v_k > 0$, $k = 0, \ldots, n$ y el conjunto de tales vectores se representa como $(\mathbb{R}^+)^{n+1}$. Los vectores cuyas entradas son todas o bien 0 o bien 1 se representarán como $\mathsf{0}_{n+1}$ y e_{n+1}, respectivamente, aunque habitualmente suprimiremos los subíndices cuando no haya lugar a confusión.

Un vector $\mathsf{z} \in \mathbb{R}^{n+1}$ se denomina *periódico de período* $p \in \mathbb{N}^*$, $1 \leq p < n$ si satisface que $z_{p+k} = z_k$, $k = 0, \ldots, n+1-p$. El subespacio de $\mathbb{R}^{n+1}$ formado por los vectores de período p se representa como $\mathbb{R}_p^{n+1}$. Cada vector en $\mathbb{R}_p^{n+1}$ está determinado por sus primeras p coordenadas: si $n+1 = mp+r$ con $m \in \mathbb{N}^*$ y $0 \leq r < p$, entonces $z_{kp+j} = z_j$, $k = 0, \ldots, m-1$ y $j = 0, \ldots, p-1$ y $z_{mp+j} = z_j$, $j = 0, \ldots, r-1$, cuando $r \geq 1$.

Consideraremos también el conjunto $(\mathbb{R}^+)_p^{n+1} = (\mathbb{R}^+)^{n+1} \cap \mathbb{R}_p^{n+1} = \{\mathsf{z} \in \mathbb{R}_p^{n+1} : \mathsf{z} > 0\}$.

Un vector $\mathsf{z} \in \mathbb{R}^{n+2}$ se denomina *casi-periódico de período* $p \in \mathbb{N}^*$, $1 \leq p < n$ isi satisface que $z_{p+k} = z_k$, for any $k = 1, \ldots, n-p$. El subespacio de vectores casi-periódicos de período p se denotará como $\mathbb{A}_p^{n+2}$. Si $n+1 = mp+r$ with $m \in \mathbb{N}^*$ y $0 \leq r < p$, entonces $\mathsf{z} \in \mathbb{A}_p^{n+2}$

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si y solo si $z_{kp+j} = z_j$, $k = 0, \ldots, m-1$ y $j = 1, \ldots, p-1$ y $z_{mp+j} = z_j$, $j = 0, \ldots, r-1$ cuando $r \geq 1$. Además, definimos las *constantes de periodicidad* $\sigma = z_0 - z_p$ y $\tau = z_{n+1} - z_r$. Es claro que si $\mathsf{z} \in \mathbb{A}_p^{n+2}$, entonces $\mathsf{z} \in \mathbb{R}_p^{n+2}$ si y solo si $\sigma = \tau = 0$.

El principal objetivo de este trabajo es describir los autovalores y autovectores correspondientes de algunas clases de matrices de Jacobi, es decir de matrices de la forma

$$\mathsf{J}(\mathsf{a},\mathsf{b}) = \begin{bmatrix} a_0 & -b_0 & 0 & \cdots & 0 \\ -b_0 & a_1 & -b_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_n & -b_n \\ 0 & \cdots & 0 & -b_n & a_{n+1} \end{bmatrix} \tag{1}$$

donde $\mathsf{a} = (a_0, a_1, \ldots, a_n, a_{n+1}) \in \mathbb{R}^{n+2}$ y $\mathsf{b} = (b_0 \ldots, b_n) \in (\mathbb{R}^+)^{n+1}$.

La matriz de Jacobi $\mathsf{J}(\mathsf{a},\mathsf{b})$ se denomina *p-Toeplitz* cuando $(\mathsf{a},\mathsf{b}) \in \mathbb{R}_p^{n+2} \times (\mathbb{R}^+)_p^{n+1}$, see for instance [1,5,6] y *casi p-Toeplitz* cuando $(\mathsf{a},\mathsf{b}) \in \mathbb{A}_p^{n+2} \times (\mathbb{R}^+)_p^{n+1}$.

Dado $\mathsf{f} \in \mathbb{R}^{n+2}$, cuando $\mathsf{J}(\mathsf{a},\mathsf{b})\mathsf{v} = \mathsf{f} \in \mathbb{R}^{n+2}$, entonces $\mathsf{v} \in \mathbb{R}^{n+2}$ es una solución del sistema lineal

$$\begin{aligned} a_0 v_0 - b_0 v_1 &= f_0, \\ -b_{k-1} v_{k-1} + a_k v_k - b_k v_{k+1} &= f_k, \quad k = 1, \ldots, n, \\ -b_n v_n + a_{n+1} v_{n+1} &= f_{n+1}. \end{aligned} \tag{2}$$

En particular, cuando $\mathsf{f} = \mathsf{0}$ el sistema anterior se denomina *homogéneo*. Aquí estamos interesados en determinar los números reales $\lambda \in \mathbb{R}$ y los vectores del subespacio

$$\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) = \{\mathsf{v} \in \mathbb{R}^{n+2} : \mathsf{J}(\mathsf{a},\mathsf{b})\mathsf{v} = \lambda \mathsf{v}\} = \ker\big(\mathsf{J}(\mathsf{a} - \lambda \mathsf{e},\mathsf{b})\big).$$

Cuando $\dim \mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) \geq 1$, λ se denomina *autovalor de* $\mathsf{J}(\mathsf{a},\mathsf{b})$ y entonces los elementos de $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda)$ se denominan *autovectores* correspondientes a λ. Es claro que los autovalores de $\mathsf{J}(\mathsf{a},\mathsf{b})$ son las raíces de $P_{\mathsf{a},\mathsf{b}}(x) = \det\big(\mathsf{J}(\mathsf{a} - x\mathsf{e},\mathsf{b})\big)$, denominado el *polinomio característico* de $\mathsf{J}(\mathsf{a},\mathsf{b})$.

Desde el punto de vista combinatorio, el sistema (2) puede interpretarse como un problema de contorno en un *camino pesado* de vértices $\{0, \ldots, n+1\}$, aristas $e_k = \{k, k+1\}$ y peso b_k en la arista e_k, $k = 0, \ldots, n$. Los vértices $\{0, n+1\}$ son habitualmente denominados *extremos del camino*, pero aquí es más conveniente referirnos a ellos como *vértices frontera*. Asimismo, los vértices $\{1, \ldots, n\}$ serán denominados *vértices interiores* del camino.

Llamamos *operador de Schrödinger discreto con coeficientes* a *y* b a $\Delta_{\mathsf{a},\mathsf{b}} \colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^n$ definido por la recurrencia de tres términos

$$\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v})_k = -b_{k-1} v_{k-1} + a_k v_k - b_k v_{k+1}, \;\; k = 1, \ldots, n, \tag{3}$$

y de esta manera la ecuación $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{f}$ sobre $\{1, \ldots, n\}$ se conoce como *ecuación de Schrödinger* con coeficientes a y b y dato f. Es claro, que en el sistema (2) las ecuaciones para los vértices interiores se corresponden con la ecuación de Schrödinger (3), pero las ecuaciones para los vértices frontera son diferentes y pueden interpretarse como *condiciones de contorno*.

Cada una de ellas involucra valores en el nodo de frontera y su único nodo adyacente, de manera que representan la versión discreta de condiciones de contorno de tipo *Sturm-Liouville*. En definitiva, el análisis del sistema (2), y por tanto de las matrices de Jacobi, puede realizarse interpretándolo como un problema de contorno para una ecuación lineal en diferencias de segundo orden. Además, como en este trabajo trataremos solo con ecuaciones homogéneas, es decir cuando $\mathsf{f} = \mathsf{0}$, en lo sucesivo la expresión ecuación de Schrödinger será usada como sinónimo de ecuación de Schrödinger homogénea.

El operador $\Delta_{\mathsf{a},\mathsf{b}}$ es independiente de las coordenadas a_0 and a_{n+1} del vector a. Clarificaremos ahora su papel considerando *operadores de contorno*, $\mathcal{B}^0_{\mathsf{a},\mathsf{b}}, \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}} \colon \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ que están definidos como

$$\mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = a_0 v_0 - b_0 v_1 \quad \text{and} \quad \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(x) = a_{n+1} v_{n+1} - b_n v_n \tag{4}$$

y que por tanto dependen de los coeficientes a_0, a_{n+1}, b_0 y b_n. Definimos ahora el *Problema de Sturm-Liouville* determinado por with a and b como el consistente en encontrar los vectores $\mathsf{v} \in \mathbb{R}^{n+2}$ satisfaciendo

$$\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{0} \quad \text{y} \quad \mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0. \tag{5}$$

Si $\mathcal{V}_{\mathsf{a},\mathsf{b}} \subset \ker(\Delta_{\mathsf{a},\mathsf{b}})$ es el subespacio de soluciones del problema (5), éste se denomina *regular* cuando $\dim \mathcal{V}_{\mathsf{a},\mathsf{b}} = 0$ y *singular* en otro caso.

El siguiente Lema caracteriza cuándo el problema (5) es singular. Todos los resultados y motivaciones pueden encontrarse en [2, 3].

Lema 1.1. *Si el problema de Sturm-Liouville* (5) *es singular, entonces* $\dim(\mathcal{V}_{\mathsf{a},\mathsf{b}}) = 1$ *y además* $\mathcal{V}_{\mathsf{a},\mathsf{b}} = \operatorname{span}\{\mathsf{v}\}$*, donde* v *es la única solución de la ecuación de Schrödinger que satisface las condiciones iniciales* $v_0 = b_0$*,* $v_1 = a_0$*.*

Después del resultado anterior, dados $\mathsf{a} \in \mathbb{R}^{n+2}$ y $\mathsf{b} \in (\mathbb{R}^+)n+1$, llamaremos *solución fundamental* de la ecuación de Schrödinger $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{z}) = \mathsf{0}$ al único $\mathsf{v} \in \mathbb{R}^{n+2}$ tal que $\Delta_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = \mathsf{0}$ y además $v_0 = b_0$ y $v_1 = a_0$, lo que implica que $\mathcal{B}^0_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0$. Así, el problema de contorno (5) es singular si y sólo si la solución fundamental de la ecuación de Schrödinger equation satisface $\mathcal{B}^{n+1}_{\mathsf{a},\mathsf{b}}(\mathsf{v}) = 0$; es decir $a_{n+1} v_{n+1} = b_n v_n$, y entonces $\mathcal{V}_{\mathsf{a},\mathsf{b}} = \operatorname{span}\{\mathsf{v}\}$.

2. Resultados

A continuación mostraremos el resultado clave sobre los autovalores y correspondientes autovectores de de la matriz de Jacobi $\mathsf{J}(\mathsf{a}, \mathsf{b})$.

Teorema 2.1. *Dado* $(\mathsf{a}, \mathsf{b}) \in \mathbb{R}^{n+2} \times (\mathbb{R}^+)^{n+1}$*, para cada* $x \in \mathbb{R}$*, consideremos* $\mathsf{v}(x)$ *la solución fundamental sde la ecuación de Schrödinder* $\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}(\mathsf{z}) = \mathsf{0}$ *en* $\{1, \ldots, n\}$*. Entonces para cada* $k = 0, \ldots, n+1$*,* $v_k(x)$ *es un polinomio en* x *de grado* k *y el polinomio característico de* $\mathsf{J}(\mathsf{a}, \mathsf{b})$ *es*

$$P_{\mathsf{a},\mathsf{b}}(x) = b_1 \cdots b_n \big((a_{n+1} - x) v_{n+1}(x) - b_n v_n(x) \big).$$

Por tanto, $\lambda \in \mathbb{R}$ *es un autovalor de* $\mathsf{J}(\mathsf{a}, \mathsf{b})$ *si y sólo si* $(a_{n+1} - \lambda) v_{n+1}(\lambda) = b_n v_n(\lambda)$ *y cuando ésto ocurre* $\mathcal{V}_{\mathsf{a},\mathsf{b}}(\lambda) = \operatorname{span}\{\mathsf{v}(\lambda)\}$.

Después del Teorema 2.1, es claro que obtener una fórmula cerrada para los autovalores de de una matriz de Jacobi es equivalente a obtener una expresión explícita para la solución fundamental de una ecuación en diferencias lineal y de segundo orden. of second order difference equations. Así pues, de manera natural surge la siguiente pregunta:

¿Para qué coeficientes $\mathsf{a} = (a_0, \dots, a_{n+1}) \in \mathbb{R}^{n+2}$ *y* $\mathsf{b} = (b_0, \dots, b_n) \in (\mathbb{R}^+)^{n+1}$ *podemos obtener explícitamente* $\mathsf{v}(x)$*, la solución fundamentalde la ecuación de Schrödinger*

$$\Delta_{\mathsf{a}-x\mathsf{e},\mathsf{b}}(\mathsf{z})_k = -b_{k-1}z_{k-1} + (a_k - x)z_k - b_k z_{k+1} = 0, \;\; k = 1, \dots, n \tag{6}$$

para cada $x \in \mathbb{R}$*?*

Recordemos que $\mathsf{v}(x)$ está caracterizada como la única solución de la ecuación (6) satisfaciendo las condiciones iniciales $v_0(x) = b_0$ y $v_1(x) = a_0 - x$. Para ello, la coordenada a_{n+1} del vector a no tiene ningún papel, solo será relevante cuando necesitemos conocer si x es un autovalor de $\mathsf{J}(\mathsf{a}, \mathsf{b})$.

Mostraremos que un escenario suficientemente general aparece cuando los coeficientes de $\Delta_{\mathsf{a},\mathsf{b}}$ son periódicos de período p; esto es, cuando $(\mathsf{a}, \mathsf{b}) \in \mathbb{A}_p^{n+2} \times (\mathbb{R}^+)_p^{n+1}$, lo que corresponde a matrices de Jacobi casi-Toeplitz. Para realizar este análisis nos basaremos en nuestros resultados sobre este tipo de ecuaciones que pueden encontrarse en [2] y también en [4] y mostraremos la cerrada relación que existe entre las soluciones fundamentales y los *polinomios de Chebyshev*. En particular, obtendremos expresiones explícitas para perídos $p = 1, 2$.

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Some results on Graphic Topology defined on Tournaments*

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Abstract. In this work, we deduce a characterization of indecomposable tournaments with few vertices. We verify that the minimum number of vertices such that there exist non-isomorphic indecomposable tournaments with homeomorphic graphic topologies is six.

1 Introduction

Several researchers have studied the problem of topologization of combinatorial structures, for example in [4] and [6]. Within the framework of topologies defined on locally finite graphs, the graphic topology was developed in [2], [5] and [1]. We continue the research about the graphic topology defined on finite indecomposable tournaments (complete digraphs), begun in [1]. This work is a starting point for the study of graphical topology first for finite tournaments with any number of vertices and then for infinite tournaments that are locally finite.

2 Preliminaries

We show some basic concepts and results on finite topological spaces and tournaments.

A topological space $(X, \mathcal{T})$ is an *Alexandroff space* if every arbitrary intersection of open sets is an open set. All finite topological space is an Alexandroff space.

If $(X, \leq)$ is a partially ordered finite set, we consider the T_0 topology on X given by the basis $\{U_x\}_{x \in X}$ where

$$U_x = \{y \in X : y \leq x\}.$$

Conversely, if $(X, \mathcal{T})$ is a finite topological space (or a finite space), the *minimal open set of* x, U_x, is the intersection of all open sets containing $x \in X$. Then we can define a preorder by saying $x \leq y$ if and only if $U_x \subset U_y$. This preorder is an order if and only if it is T_0. Subsequently, a bijective correspondence can be established between finite T_0-spaces and finite partially ordered sets (finite posets). This correspondence allows us to interpret some topological properties of finite spaces in terms of the order relation, such as connectivity, continuity, ... For example, a map between finite T_0-spaces is continuous if and only if it is order preserving.

We recall that a *tournament* is an oriented complete graph, i. e., a directed graph or digraph in which any pair of vertices is connected by only one directed edge.

Given a tournament $T = (V, A)$ and $x, y \in V$, we say that $\mathbf{x}$ *dominates* $\mathbf{y}$, $x \to y$, if there is a directed edge from x to y, i. e. $(x, y) \in A$. A subset $I \subseteq V$ is an *interval* if $\forall a, b \in I$,

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$\forall x \in V \setminus I,\ a \to x$ if and only if $b \to x$. The *trivial intervals* of a tournament are the subsets $\emptyset$, V and $\{x\}, \forall x \in V$.

Definition 1. *A tournament is **indecomposable** if all its intervals are trivial. Otherwise, the tournament is called **decomposable**.*

A *path in* a tournament is a sequence of vertices $x = z_0, z_1, \ldots, z_n = y$ such that $\forall i \in \{0, \ldots, n-1\},\ z_i \to z_{i+1}$. A finite tournament $T = (V, A)$ of three or more vertices is *strongly connected* if $\exists\, a \in V$ so that $\forall\, v \in V$, $v \neq a$, there exist paths in T from a to v and from v to a.

Proposition 1. *A tournament of three or more vertices is strongly connected if and only if it is indecomposable.*

We use the following notation for any tournament $T = (V, A)$ and any vertex x of T:

Outset: $N_x^+ = \{y \in V : x \to y\}$

Inset: $N_x^- = \{y \in V : y \to x\}$

Out-degree: $d^+(x) = |N_x^+|$

In-degree: $d^-(x) = |N_x^-|$

Now we can define the graphic topology on tournaments.

Definition 2. *The family*

$$\mathcal{S}_T = \{N_x^+ : x \in V\}$$

*is a subbasis for a topology $\mathcal{T}_T$ on V, called the **graphic topology** of T.*

It is easy to check the next result.

Lemma 1. *For all $x \in V$, $U_x = \bigcap_{y \in N_x^-} N_y^+$.*

Although in this work we consider finite tournaments, by the last lemma it is possible to define the graphical topology on infinite tournaments that are locally finite, i.e., those with finite maximum in-degree.

Proposition 2. *Let $T = (V, A)$ be a finite indecomposable tournament, $|V| \geq 3$. Then $(V, \mathcal{T}_T)$ is a finite T_0-space.*

3 Results

We use the representation of the finite tournaments following by W. Moon [3] in the classification of non-isomorphic tournaments with at most six vertices:

- If there is no edge between two vertices, the one on top dominates the one beneath it.
- Unlabeled vertices are named in alphabetic order from top to bottom and if there is a directed cycle, in the direction of the cycle starting from the top-left vertex.

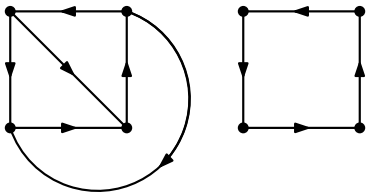

Figure 1: A tournament with four vertices and its representation.

Proposition 3. *A tournament of n vertices, with $3 \leq n \leq 6$, is decomposable if and only if its representation verifies one of these properties:*

(1) There is a vertex v over or bellow the rest of the representation, without incident edges.
(2) There is a directed cycle represented, with fewer than n vertices, that does not have any exterior incident edges.
(3) There are two points, one bellow the other one, without exterior incident edges.

The decomposable tournaments in Figure 2 verify properties *(1)*, *(2)* and *(3)* from the previous proposition, respectively.

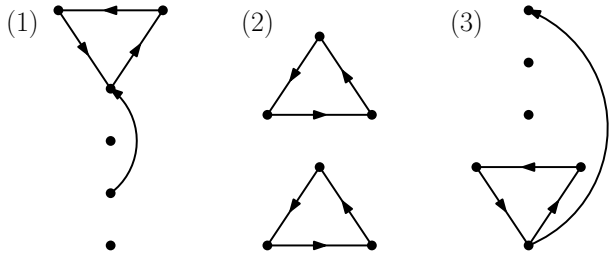

Figure 2: Decomposable tournaments.

Proposition 4. *There exist finite T_0-spaces of six vertices that cannot be obtained from the graphic topology of an indecomposable tournament.*

Theorem 1. *There are 6 families of homeomorphic tournaments of 6 vertices.*

The indecomposable tournaments in Figure 3 form one of these families. It can be easily deduced that its graphic topologies are homeomorphic (the graphic topologies are represented bellow the tournaments as posets).

4 Conclusions and future work

In this work, we have characterized indecomposable tournaments with three to six vertices and obtained the graphic topologies of the non-isomorphic ones. We have also obtained that there are no tournaments of three to five vertices, indecomposable and non-isomorphic, whose graphical topologies are homeomorphic, but this situation is possible for six vertices. In addition, we have proven that the family of graphic topologies of non-isomorphic indecomposable tournaments of six vertices is strictly included in the family of finite T_0-spaces of six vertices.

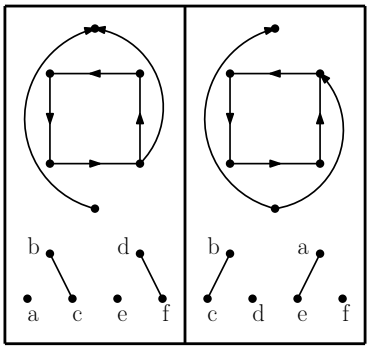

Figure 3: Two non-isomorphic indecomposable tournaments with homeomorphic graphic topologies.

Finally, we propose some open problems that arise from the results obtained.

- Characterize indecomposable tournaments with an arbitrary number of vertices.
- Study properties of the graphic topology defined on finite indecomposable tournaments.
- Identify characteristics of non-isomorphic indecomposable tournaments with homeomorphic graphic topologies.
- Determine how large the set of finite T_0-spaces that cannot be obtained as the graphical topology of an indecomposable tournament is by increasing the number of vertices.

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ROUDNEFF'S CONJECTURE IN DIMENSION 4*

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Abstract. J.-P. Roudneff conjectured in 1991 that every arrangement of $n \geq 2d+1 \geq 5$ (pseudo) hyperplanes in the real projective space $\mathbb{P}^d$ has at most $\sum_{i=0}^{d-2} \binom{n-1}{i}$ cells bounded by each hyperplane. The conjecture is true for $d = 2, 3$. The main result of this manuscript is to show the validity of this conjecture for $d = 4$.

1 Introduction

An *Euclidean* (resp. *projective*) d-arrangement of n hyperplanes $H(d,n)$ is a finite collection of hyperplanes in the Euclidean space $\mathbb{R}^d$ (resp. the real projective space $\mathbb{P}^d$) such that no point belongs to every hyperplane of $H(d,n)$. Any arrangement $H(d,n)$ decomposes $\mathbb{R}^d$ (resp. $\mathbb{P}^d$) into a d-dimensional cell complex. It is clear that any d-cell c of $H(d,n)$ has at most n *facets* (that is, $(d-1)$-cells). We say that c is a *complete cell* of $H(d,n)$ if c has exactly n facets, i.e., c is bounded by each hyperplane of $H(d,n)$.

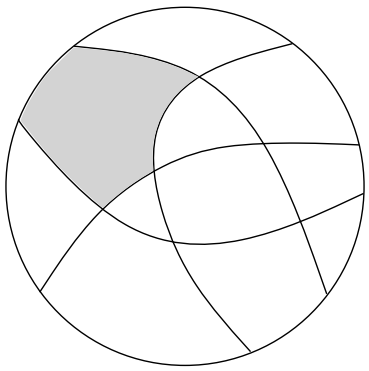

Figure 1: An arrangement of 5 hyperplanes in $\mathbb{P}^2$. The gray cell is a complete cell.

The *cyclic polytope* of dimension d with n vertices, discovered by Carathéodory [2], is the convex hull in $\mathbb{R}^d$ of $n \geq d+1 \geq 3$ different points $x(t_1), \ldots, x(t_n)$ of the moment curve $x : \mathbb{R} \longrightarrow \mathbb{R}^d,\ t \mapsto (t, t^2, \ldots, t^d)$. Cyclic polytopes play an important role in the combinatorial convex geometry due to their connection with certain extremal problems, For example, the

*(Acknowledgments)

Upper Bound theorem due to McMullen [8]. *Cyclic arrangements* are defined as the dual of the cyclic polytopes. As for cyclic polytopes, cyclic arrangements also have extremal properties. For instance, Shannon [9] has introduced cyclic arrangements on dimension d as examples of projective arrangements with a minimum number of cells with $(d+1)$ facets.

Let denote by $C_d(n)$ the number of complete cells of the cyclic arrangements on dimension d with n hyperplanes. Roudneff [6] proved that $C_d(n) \geq \sum_{i=0}^{d-2} \binom{n-1}{i}$ and that is tight for all $n \geq 2d+1$. Moreover, he conjectured that in that case, cyclic arrangements have the maximum number of complete cells.

Conjecture 1.1. *[6, Conjecture 2.2] Every arrangement of $n \geq 2d+1 \geq 5$ (pseudo) hyperplanes in $\mathbb{P}^d$ has at most $C_d(n)$ complete cells.*

The conjecture is true for $d = 2$ (that is, any arrangement of n pseudolines in $\mathbb{P}^2$ contains at most one complete cell), Ramírez Alfonsín [5] proved the case $d = 3$ and in [7] the authors proved it for arrangements arising from Lawrence oriented matroids.

In [3] calculated the exact number of complete cells of cyclic arrangements for any positive integers d and n with $n \geq d+1$, namely,

$$C_d(n) = \binom{d}{n-d} + \binom{d-1}{n-d-1} + \sum_{i=0}^{d-2} \binom{n-1}{i}. \tag{1}$$

Thus, in view of Roudneff's conjecture, Montejano and Ramírez Alfonsín [7] asked the following.

Question 1.2. *Is it true that every (pseudo) arrangement of $n \geq d+1 \geq 3$ hyperplanes in $\mathbb{P}^d$ has at most $C_d(n)$ complete cells?*

The main result of this work is to answer affirmatively to Question 1.2 for $d = 4$. As a consequence, we prove Roudneff's conjecture for dimension 4, giving more credit to the general conjecture. Moreover, with some simple observations, we can finish answering to Question 1.2 for $d = 3$.

2 Results

Many of the combinatorial properties of arrangements of (pseudo) hyperplanes can be studied in the language of oriented matroids. The so-called Topological Representation Theorem, due to Folkman and Lawrence [4], states that the acyclic reorientation classes of oriented matroids on n elements and rank r (without loops or parallel elements) are in one-to-one correspondence with the classes of isomorphism of arrangements of n (pseudo) hyperplanes in $\mathbb{P}^{r-1}$.

An arrangement $H(d,n)$ is called *simple* if $n \geq d$ and every intersection of d pseudo-hyperplanes is a unique distinct point. It is known that simple arrangements correspond to uniform oriented matroids. The d-cells of any arrangement $H(d,n)$ are usually called topes since they are in one-to-one correspondence with the topes of the oriented matroids M on n

elements of rank $r = d + 1$ of its corresponding acyclic reorientation class. It is known that a tope is a complete cell if reorienting any single element, the resulting sign-vector is also a tope.

Theorem 2.1. *Each of the* 135 *acyclic reorientation classes of uniform rank* 5 *oriented matroids on* 8 *elements have at most* $2C_4(8)$ *complete cells. Moreover, there is only* 1 *acyclic reorientation class with exactly* $2C_4(8)$ *complete cells.*

Theorem 2.2. *Each of the* $9\ 276\ 595$ *acyclic reorientation classes of uniform rank* 5 *oriented matroids on* 9 *elements have at most* $2C_4(9)$ *complete cells. Moreover, the class of the alternating oriented matroid is the only one with exactly* $2C_4(9)$ *complete cells.*

Theorem 2.3. *Every arrangement of* $n \geq 5$ *(pseudo) hyperplanes in* $\mathbb{P}^4$ *has at most* $C_4(n)$ *complete cells.*

3 Conclusions

Until now Roudneff's conjecture has been verified for dimensions $d = 2, 3, 4$ and for arrangements arising from Lawrence oriented matroids. To prove Roudneff's conjecture for dimension 4, we used some ideas and techniques taken from oriented matroids.

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Coloraciones por entornos en jaulas de Moore*

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Abstract. A k-coloring of a graph G is a k-partition $\Pi = \{S_1, ..., S_k\}$ of $V(G)$ into independent sets, called colors. A k-coloring is called *neighbor-locating* if for every pair of vertices $\{u, v\}$ belonging to the same color S_i, the set of colors of the neighborhood of u is different from the set of colors of the neighborhood of v. The *neighbor-locating chromatic number*, denoted by $\chi_{NL}(G)$, is the minimum cardinality of a neighbor-locating coloring of G. In this paper we study the neighbor-locating coloring of graphs that are Moore cages.

1. Introduccion

Una k-coloración de un grafo G es una k-partición $\Pi = \{S_1, ..., S_k\}$ de $V(G)$ en conjuntos independientes, llamados colores. Una k-coloración se dice que *resuelve por vecindades* (o que es una χ_{NL}-partición, para abreviar) si para cada par de vértices $\{u, v\}$ del mismo color S_i, el conjunto de colores de los vecinos de u es diferente del conjunto de colores de vecinos de v. El *número cromático resolviendo por vecindades*, $\chi_{NL}(G)$, es el mínimo cardinal de una una χ_{NL}-partición de G.

Este tipo de coloraciones fueron introducidas de forma independiente en [4] y [1]. Se conocen cotas ajustadas de $\chi_{NL}(G)$ [1, 2, 5]. Las χ_{NL}-coloraciones han sido estudiadas en varias familias de grafos: caminos [2], ciclos [2], grafos completos [1], grafos de Minkowski [1], en algunos grafos de Jhonson [1] y en grafos dispersos [6]. También ha sido estudiado el comportamiento de $\chi_{NL}(G)$ con algunos tipos de operaciones de grafos [7, 8]. En este trabajo vamos a estudiar estas nociones en ciertos grafos de Moore.

2. Definiciones y resultados previos

Un (r, g)-grafo se define como un grafo en el cual cada vértice tiene exactamente r vecinos, y en el cual el ciclo más corto tiene longitud exáctamente g. A g se le llama *cintura* del grafo. Se sabe que existen (r, g)-grafos para cualquier combinación de $r \geq 2$ y $g \geq 3$. Una (r, g)-jaula es un (r, g)-grafo con el menor número de vértices posible, entre todos los (r, g)-grafos. El mínimo número de vértices, n, que puede tener un (r, g)-grafo se conoce como *cota de Moore* y relaciona n con el grado r y el diámetro D del grafo, concretamente:

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$$n = 1 + r\sum_{i=0}^{D}(r-1)^i.$$

Las jaulas que alcanzan esta cota se les llama *jaulas de Moore* (ver [3]).

Algunas jaulas de Moore son las siguientes:

- los ciclos de orden n, C_n, son $(2, n)$-jaulas de Moore.
- los grafos completos de orden n, K_n, son $(n-1, 3)$-jaulas de Moore.
- los grafos bipartitos completos de orden $n = 2r$, $K_{r,r}$ son $(r, 4)$-jaulas de Moore.

El número cromático resolviendo por vecindades de estas jaulas de Moore los conocemos por trabajo realizado en [1] y [2]. Concretamente se obtienen los siguientes resultados.

Teorema 2.1. *Se verifica:*

- $\chi_{NL}(K_n) = n$
- $\chi_{NL}(K_{r,r}) = 2r = n$
- $\chi_{NL}(C_n) = 3$ *si* $n \in \{3, 5, 7, 9\}$.
- $\chi_{NL}(C_n) = 4$ *si* $n \in \{4, 6, 8\}$.
- *Sea* $k \geq 4$, $\ell(k) = \frac{k^3-k^2}{2}$ *y* $\ell(k-1) \leq n \leq \ell(k)$, *entonces:*
  - $\chi_{NL}(C_n) = k$ *si* $n \neq \ell(k) - 1$.
  - $\chi_{NL}(C_n) = k + 1$ *si* $n = \ell(k) - 1$.

En algunos resultados de la siguiente sección utilizamos las siguientes cotas del orden del grafo G en función del grado máximo y de $\chi_{_{NL}}(G)$ calculadas en [1].

Teorema 2.2. *Sea G grafo conexo de orden $n \geq 2$, grado máximo Δ y $\chi_{_{NL}}(G) = k$. Entonces,*

$$n \leq min\{k\,(2^{k-1} - 1),\ \sum_{j=1}^{\Delta}\binom{k-1}{j}\}.$$

3. Número cromático resolviendo por vecindades en jaulas

Empezamos por dos grafos muy conocidos. En el primero ha sido fácil ver que justo se alcanza la cota dada al final de la sección anterior. Más trabajo ha dado demostrar que no se puede $X_{_{NL}}$-colorear el segundo grafo con 4 colores. Para esta prueba se han utilizado las propiedades del grafo H como grafo intersección de un plano proyectivo.

Teorema 3.1. *Se verifica:*

Si G es la $(3, 5)$-jaula de Moore o grafo de Petersen, entonces $\chi_{NL}(G) = 4$.

Si H es la $(3, 6)$-jaula de Moore o grafo de Heawood, entonces $\chi_{NL}(H) = 5$.

Centramos ahora nuestro estudio en las jaulas de Moore de cuello 6. Hemos obtenido la siguiente cota superior.

Teorema 3.2. *Sea G una $(r, 6)$-jaula de Moore, siendo $r-1$ un número primo, entonces:*

$$\chi_{NL}(G) \leq 2r - 1.$$

Para ver que esta cota es ajustada basta observar el valor obtenido en el grafo de Heawood. En cuanto a la cota inferior, estamos trabajando en la demostración del siguiente resultado.

Conjetura 3.3. *Sea G una $(r, 6)$-jaula de Moore, siendo $r-1$ un número primo, entonces:*

$$r + 2 \leq \chi_{NL}(G).$$

Observamos que lo conjeturado se ajusta al resultado obtenido en el grafo de Heawood.

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Estudio de Caminos Aleatorios con M-matrices*

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Resumen. En el ámbito del los caminos aleatorios, el tiempo medio de paso y la constante de Kemeny nos permiten profundizar en el estudio de redes. Para una matriz de transición de probabilidad P, en la literatura podemos observar como los autores caracterizan el tiempo medio de paso usando inversas generalizadas de $\mathsf{I}-\mathsf{P}$, y específicamente con su inversa de grupo. En este trabajo analizamos el comportamiento de los caminos aleatorios en redes usando la matriz asociada al Laplaciano combinatorio, aprovechando que esta matriz es una M-matriz. De esta forma, conseguimos fórmulas cerradas para la matriz del tiempo medio de paso y para la constante de Kemeny's basadas en la inversa de grupo del Laplaciano combinatorio.

1. Introducción

Los caminos aleatorios representan un buen modelo de sistemas con elementos que interactuan entre ellos. Esta interacción puede modelarse con un cierto grafo o red, asignando a cada arista una cierta probabilidad de transición. Por tanto, el movimiento a través del grafo puede ser representado por un camino aleatorio tal que si nos colocamos al inicio en un cierto vértice (o nodo), nos movemos a continuación a uno de sus vértices vecinos de forma aleatoria. De esa manera, podemos referirnos a caminos aleatorios sobre grafos [8].

La *matriz de transición* P proporciona las probabilidades de las transiciones entre nodos en un camino aleatorio. Así que es posible estudiar su comportamiento a largo y corto plazo. Es bien conocido que para el comportamiento a largo plazo tenemos que considerar un autovector de P, que se denota por $\boldsymbol{\pi}$ y se denomina *distribución estacionaria*. Para el comportamiento a corto plazo se define el *tiempo medio de paso* (MFPT en sus siglas en inglés) del nodo i al nodo j, es decir, el número esperado de pasos para alcanzar por primera vez j desde el nodo inicial i. Además, el número de pasos esperado para alcanzar aleatoriamente cualquier vértice i según $\boldsymbol{\pi}$ es una constante, independientemente del vértice de partida. Este valor es conocido como *constante de Kemeny*. Remitimos al lector a la referencia [7] para un estudio mas detallado de estos dos conceptos. Existe una amplia literatura en el estudio de la constante de Kemeny y del MFPT que relaciona estos parámetros con inversas generalizadas de $\mathsf{I}-\mathsf{P}$, veáse por ejemplo [5, 6].

Nuestro propósito es describir el MFPT y la constante de Kemeny usando inversas generalizadas asociadas con el Laplaciano combinatorio en vez 1-inversas de la matriz $\mathsf{I}-\mathsf{P}$, puesto

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que el Laplaciano combinatorio es una Z-matrix simétrica y semidefinida positiva, es decir, una M-matriz y, por tanto, podemos sacar rédito de sus propiedades, veáse [1, 2].

2. Caminos aleatorios y el Laplaciano combinatorio

Sea $\Gamma = (V, E, c)$ una red; es decir, un grafo finito y conexo sin lazos ni aristas múltiples, con un conjunto de vértices V y conjunto de aristas E, siendo $c \in \mathbb{R}^+$ una conductancia asignada a cada vértice. Los cardinales de los dos conjuntos son $|V| = n$ y $|E| = m$, respectivamente. Decimos que i es adyacente a j, $i \sim j$, si $\{i, j\} \in E$ y, en ese caso, asignamos a la arista una *conductancia* $c_{ij} > 0$; en otro caso, $c_{ij} = 0$. Entonces, definimos $k_i = \sum_{j \in V} c_{ij}$ como el *grado* de i. Cuando $c_{ij} = 1$ para cualquier $i \sim j$, Γ se denomina *grafo*. En ese caso, k_i es el número de vértices adyacentes a i, es decir, el grado usual en un grafo.

Si etiquetamos el conjunto de vértices V, podemos identificar el Laplaciano combinatorio con la matriz simétrica irreducible

$$\mathsf{L} = \begin{bmatrix} k_1 & -c_{12} & \dots & -c_{1n} \\ -c_{12} & k_2 & \dots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \dots & k_n \end{bmatrix}.$$

Esta matriz es diagonalmente dominante y, entonces, es semidefinida positiva. Además, es singular y 0 es un autovalor simple cuyo autovector asociado es constante, $\mathsf{L}\mathbf{1} = \mathbf{0}$, donde $\mathbf{1}$ es el vector de unos.

Para comodidad del lector, los vectores aparecen con notación en negrita. En particular, $\mathbf{k} = \big(k_1, k_2, \dots, k_n\big)^{\mathsf{T}}$ es el vector de grados para Γ. Además, dada una matriz A y un vector $\mathbf{v}$, denotamos por A_d la matriz diagonal cuyos elementos diagonales son a_{ii} y por $\mathsf{D}_{\mathbf{v}}$ la matriz diagonal cuyos elementos diagonales vienen dados por los elementos de $\mathbf{v}$.

2.1. El tiempo medio de paso

El comportamiento a corto plazo de un camino aleatorio se modela por el *tiempo medio de paso* m_{ij}, para $i, j = 1, \dots, n$, $i \neq j$, que proporciona el número esperado de pasos $t \geq 1$, antes de que el sistema alcance j si empieza en i,

$$m_{ij} = E\big[t \mid X_t = j, X_0 = i\big],$$

donde $E[\bullet]$ denota la esperanza de $\bullet$. Es bien conocido [7] que, para $i \neq j$, $1 \leq i, j \leq n$,

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}\big(m_{kj} + 1\big) = 1 + \sum_{k \neq j} p_{ik} m_{kj}. \tag{1}$$

Paralelamente, el *tiempo medio de recurrencia del vértice* i, denotado por m_{ii}, es el número esperado de pasos antes de retornar a i por primera vez, para cualquier $i = 1, \dots, n$. El tiempo medio de recurrencia de i también verifica la Ecuación (1). Si definimios J como la matriz de orden n con todas las entradas iguales a 1, podemos escribir (1) en forma matricial igual que en [3],

$$\mathsf{M} = \mathsf{J} + \mathsf{PM} - \mathsf{PM}_d. \tag{2}$$

Entonces, $m_{ii} = \dfrac{1}{\pi_i}$, puesto que multiplicando ambos lados de (2) por $\boldsymbol{\pi}^{\mathsf{T}}$, obtenemos $\mathbf{0}^{\mathsf{T}} = \boldsymbol{\pi}^{\mathsf{T}}(\mathsf{J} - \mathsf{PM}_d)$ o bien $\mathbf{0}^{\mathsf{T}} = \mathbf{1}^{\mathsf{T}} - \boldsymbol{\pi}^{\mathsf{T}}\mathsf{M}_d$.

Podemos usar esta última expresión y la Ecuación (2) para obtener la expresión matricial del MFPT

$$(\mathsf{I} - \mathsf{P})\mathsf{M} = \mathsf{J} - \mathsf{PD}_{\boldsymbol{\pi}}^{-1}. \tag{3}$$

El sistema (3) tiene solución porque cada columna de $\mathsf{J} - \mathsf{PD}_{\boldsymbol{\pi}}^{-1}$ pertenece a $\boldsymbol{\pi}^{\perp}$ y la solución es única salvo por un múltiplo de $\boldsymbol{\pi}$. Usualmente, se utilizan inversas generalizadas de $\mathsf{I} - \mathsf{P}$ para resolver el sistema anterior, veáse por ejemplo [4]. Recordemos que si A es cualquier matriz singular de orden $m \times n$, una inversa generalizada o 1-inversa de A es cualquier matriz X tal que $\mathsf{AXA} = \mathsf{A}$. Para cualquier 1-inversa $\widetilde{\mathsf{G}}$ de $\mathsf{I} - \mathsf{P}$, obtenemos que

$$\mathsf{M} = \widetilde{\mathsf{G}}\big(\mathsf{J} - \mathsf{PD}_{\boldsymbol{\pi}}^{-1}\big) + \mathbf{1}\boldsymbol{\alpha}^{\mathsf{T}},$$

siendo $\boldsymbol{\alpha}$ un vector constante.

Mostramos a continuación una expresión del MFPT en términos de una 1-inversa del Laplaciano L, en vez de utilizar 1-inversas de $\mathsf{I} - \mathsf{P}$.

Proposición 2.1. *Sea Γ una red conexa y G una 1-inversa de L, entonces la matriz del tiempo medio de paso M puede escribirse como*

$$\mathsf{M} = \mathsf{GD}_{\mathbf{k}}\mathsf{J} - \mathsf{J}\big(\mathsf{GD}_{\mathbf{k}}\mathsf{J}\big)_d + \mathrm{vol}(\Gamma)\Big(\mathsf{D}_{\mathbf{k}}{}^{-1} - \mathsf{G} + \mathsf{JG}_d\Big).$$

2.2. La constante de Kemeny

El famoso parámetro $K \equiv \sum_{j \in V} m_{ij}\pi_j$ representa el número de pasos en alcanzar un vértice aleatorio j iniciando el camino en el nodo i, según la distribución estacionaria $\boldsymbol{\pi}$. Se da el hecho curioso que K no depende de i, y por eso la denominación de *constante de Kemeny*. En forma matricial, se escribe como $\mathsf{M}\boldsymbol{\pi} = K\mathbf{1}$.

Nuestro propósito ahora es expresar la constante de Kemeny usando G, una 1-inversa del Laplaciano combinatorio.

Proposición 2.2. *Si G es una 1-inversa de L tal que $\mathsf{Gk} = g\mathbf{1}$, la constante de Kemeny viene dada por*

$$K = 1 - g + \mathrm{tr}(\mathsf{GD}_{\mathbf{k}}).$$

3. Conclusiones

En este trabajo proporcionamos expresiones alternativas para los parámetros fundamentales en el marco de los caminos aleatorios que involucran inversas generalizadas del Laplaciano combinatorio, que es M-matriz simétrica y singular.

Los resultados obtenidos son equivalentes a los obtenidos por Hunter en su gran producción (veáse [5] como ejemplo) o a trabajos similares de otros autores en [9–11].

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The weak k-metric dimension of Hamming graphs

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Abstract. The k-metric dimension of a graph G is the cardinality of a smallest set of vertices X such that any two vertices of G have different distances to at least k vertices from X. The weak k-metric dimension of a graph G is a version of the k-metric dimension, defined as the cardinality of a smallest set of vertices S such that the sum of the distance differences from any pair of vertices to all vertices of S is at least k. The largest integer k such that G contains a weak k-resolving set is denoted by $\kappa(G)$. In this work, we first compute the value $\kappa(G)$ when G is a Hamming graph, and next find the weak k-metric dimension of such graphs G for every integer $k \in \{1, \ldots, \kappa(G)\}$.

1 Introduction

Topics concerning metric dimension related parameters in graphs are very well known in the research community, in connection with its applicability to diverse practical problems of nodes identification in networks. One can find in the literature several studies dealing with this topic in different directions, including theoretical combinatorial, as well as, computational and applied points of view.

One of the most common research line on this subject concerns studying different instances of the metric dimension in graphs which have had their birth on the classical metric dimension concept introduced independently in [3, 6]. The reason is to give more insight into the classical variant, or to better understand some practical situations in which extra properties are needed. For more information on the classical concept and related ones, we refer to the two surveys [4,7]. The weak k-metric dimension was recently introduced in [5], as a stronger variation than the classical metric dimension, and less restrictive than the one called k-metric dimension, already known from [2].

We now begin to formalize all the required notations and terminologies that shall be used here. To this end, throughout our whole work $G = (V(G), E(G))$ represents a connected simple graph. Given three vertices $x, y, z \in V(G)$, it is said that

$$\Delta_z(x, y) = |d(x, z) - d(y, z)|,$$

where $d(a,b)$ is the length of a shortest a,b-path in G, i.e., the distance between a and b. Consider a set $S \subseteq V(G)$ and an integer $k \geq 1$. The set S is a *weak k-resolving set* for G if it is satisfied that

$$\sum_{w \in S} \Delta_w(x,y) \geq k$$

for each two vertices $x, y \in V(G)$. The *weak k-metric dimension* of G, denoted by $\mathrm{wdim}_k(G)$, is the cardinality of a smallest weak k-resolving set of G. Moreover, any weak k-resolving set having cardinality equal to $\mathrm{wdim}_k(G)$ is called a *weak k-metric basis* for G. It might be noticed that the case $k = 1$ coincides precisely with the classical concept of metric dimension.

Notice that a graph G does not have weak k-resolving sets for every integer k. Then, $\kappa(G)$ is the largest integer k such that G contains a weak k-resolving set. These concepts above were first presented in [5].

This work is devoted to compute the weak k-metric dimension of the Cartesian product of two complete graphs $K_n \,\square\, K_n$, also called 2-dimensional Hamming graphs, for those suitable values of k. To this end, we were first required to compute $\kappa(G)$ for all 2-dimensional Hamming graphs $G = K_n \,\square\, K_n$.

2 Results

In this section we first recall some fundamental fact. For that we need some extra notation. If x and y two different vertices of G and $S = V(G)$, then let

$$\Delta(x,y) = \sum_{s \in V(G)} \Delta_s(x,y)\,.$$

Proposition 2.1. [5, Observation 5] *If G is a graph, then*

$$\kappa(G) = \min\{\Delta(x,y) :\ x,y \in V(G), x \neq y\}\,.$$

The proposition above is used to prove the following result.

Theorem 2.2. *If $r \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 2$, then*

$$\kappa(K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_r}) = 2n_2 \cdots n_r\,.$$

In particular, if $n_1 = 2$, then $\kappa(Q_r) = 2^r$.

Since the case $k = 1$ coincides precisely with the classical concept of metric dimension, and the metric dimension of $K_n \,\square\, K_n$ was already studied in [1], we centered our attention in the cases $k \geq 2$. Moreover, in order to obtain the results of our work, we needed to separate the deductions into two cases. The first one of them was for the situation $k = 2$. Further on, we managed to consider all the remaining cases together.

For the case $k = 2$, we obtain the next result.

Theorem 2.3. *If $n \geq 3$, then* $\mathrm{wdim}_2(K_n \,\square\, K_n) = \left\lceil \frac{4n}{3} \right\rceil$.

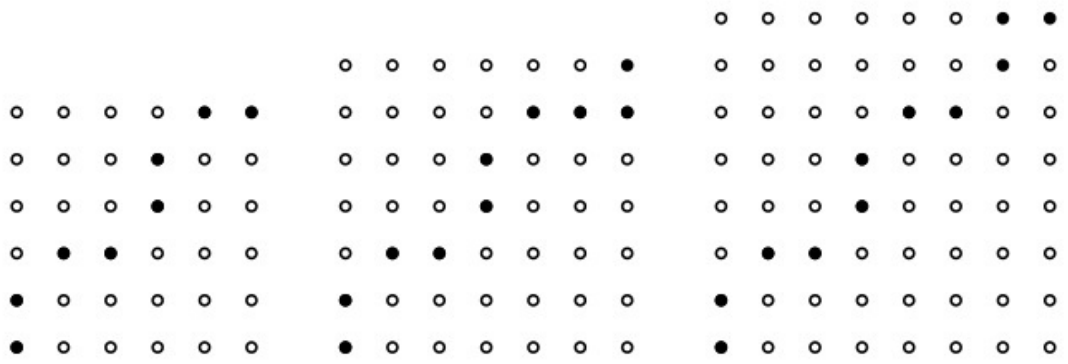

Figure 1: The bold vertices form the weak 2-metric basis respectively in $K_6 \,\square\, K_6$, $K_7 \,\square\, K_7$, and $K_8 \,\square\, K_8$.

Some examples of the structures that have the weak 2-resolving sets of $K_n \,\square\, K_n$ are shown in Figure 1.

As mentioned, the remaining values of k, that is when $k \in \{3, \ldots, \kappa(K_n \,\square\, K_n)\}$ were treated together, and we obtained the following result.

Theorem 2.4. *Let n, t be two integers with $n \geq 3$ and $0 \leq t \leq n-2$. If $3 \leq k \leq 2n$, then*

$$\mathrm{wdim}_k(K_n \,\square\, K_n) = \begin{cases} n^2 - tn; & k = 2n - 2t\,, \\ n^2 - tn - 1; & k = 2n - 2t - 1\,. \end{cases}$$

Once obtained these results, one might think to consider studying the weak k-metric dimension of n-dimensional Hamming graphs in general for $n \geq 3$, and emphasizing in the special case of hypercubes Q_n, for which we only know the value of $\kappa(Q_n)$.

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Desigualdad isoperimétrica lineal, p-parabolicidad y métricas doblantes en grafos

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Resumen. Dado un grafo métrico, estudiamos la relación entre tres propiedades: ser doblante, ser p-parabólico y satisfacer la desigualdad isoperimétrica lineal (o de Cheeger). Probamos que si un grafo uniforme G satisface la desigualdad isoperimétrica lineal, entonces G no es doblante mientras que el recíproco no es cierto. También probamos que si G es un grafo doblante con constante C, entonces es p-parabólico para todo $p \geq \log_2(C)$ y vemos que el recíproco no es cierto. Además, vemos que ser doblante no implica ser p-parabólico para todo $p \in (1, \infty)$. Finalmente, probamos que si un grafo uniforme es hiperbólico (en el sentido de Gromov) y tiene desigualdad isoperimétrica lineal, entonces no es p-parabólico para ningún $p \in (1, \infty)$.

1. Introducción

Consideramos un grafo métrico donde cada arista tiene longitud 1 y la métrica de longitud usual donde la distancia entre dos puntos es la longitud del camino mínimo que los une. Decimos que el grafo es *uniforme* si hay una cota superior finita para el número de aristas incidente en cualquier vértice.

Se dice que el grafo es *doblante* si hay una constante C tal que dado cualquier R, toda bola de radio $2R$ se puede recubrir por, a lo sumo, C bolas de radio R.

La constante de Cheeger de un grafo G se define como

$$h(G) = \inf_A \frac{|\partial A|}{|A|},$$

donde A recorre todos los subconjuntos de vértices finitos no vacíos, $\partial A = \{v \in G : d_G(v, A) = 1\}$ y $|A|$ denota el cardinal de A.

Se dice que un grafo G *satisface la desigualdad isoperimétrica de Cheeger* (o la desigualdad isoperimétrica lineal) si $h(G) > 0$, dado que en este caso

$$|A| \leq h(G)^{-1} |\partial A|, \tag{1}$$

para todo conjunto finito $A \subseteq V(G)$.

Un grafo G es p-parabólico si todas las funciones p-superharmónicas sobre G son constantes. Esta propiedad se puede caracterizar a través de la p-capacidad, como establece el Teorema 1.1 a continuación.

Dada una función u sobre un grafo G, se define el *p-módulo de su gradiente discreto*, $|\nabla_G u|_p$, y su *p-integral de Dirichlet discreta*, $D_{p,G}(u)$, respectivamente, como

$$|\nabla_G u|_p(x) := \Big(\sum_{y\in N(x)} \big|u(y)-u(x)\big|^p \Big)^{1/p}, \qquad D_{p,G}(u) := \sum_{x\in G} |\nabla_G u|_p^p(x) = 2 \sum_{vw\in E(G)} \big|u(v)-u(w)\big|^p,$$

donde las aristas se consideran sin orientación.

Dado un subconjunto finito S de G, la *p-capacidad* de S se define como

$$\operatorname{cap}_p S = \operatorname{cap}_p(S,G) = \text{ínf} \Big\{ D_{p,G}(u) : u \text{ función sobre } G \text{ con soporte finito, } u|_S = 1 \Big\}.$$

El siguiente resultado (ver [2, proposición 6] y [1, nota 5.16]) relaciona la p-capacidad y la p-parabolicidad.

Teorema 1.1. *Dado $1 < p < \infty$, un grafo uniforme G es p-parabólico si y solo si $\operatorname{cap}_p S = 0$ para algún (o, equivalentemente, para todo) subespacio finito no vacío $S \subset G$.*

2. Resultados

Teorema 2.1. *[4] Si un grafo uniforme G satisface la desigualdad isoperimétrica lineal (i.e. de Cheeger), entonces G no es doblante.*

Teorema 2.2. *[4] Si G es un grafo doblante con constante C, entonces es p-parabólico para todo $p \geq \log_2(C)$.*

Proposición 2.3. *[4] $\mathbb{Z}^n$ es un grafo doblante que no es p-parabólico para ningún $p \in (1, n)$.*

Un grafo métrico es *hiperbólico* (en el sentido de Gromov) si existe una constante $\delta > 0$ tal que todos los triángulos (entendiendo por triángulo tres puntos cualesquiera conectados por geodésicas) son δ-finos, es decir, todo punto del triángulo está a distancia menor que δ de los otros dos lados.

Teorema 2.4. *[3] Dado un grafo hiperbólico G, si $h(G) > 0$, entonces G no es p-parabólico para ningún $p \in (1, \infty)$.*

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A medical application of a discrete inverse problem

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Abstract. The so called inverse problem is one of the major mathematical challenge of our time. Essentially, it involves determining the underlying causes or parameters based on the observations of their effects. These types of problems arise in a wide range of fields.

The talk will address three fundamental aspects of the inverse problem: the mathematical formulation, numerical techniques for its resolution, and a practical application to the field of medicine.

The objective of the talk is to provide a detailed understanding of the current project where our research group is nowadays committed.

Resolución de sistemas de ecuaciones sobre el semianillo $\min - \max$

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Abstract. El proposito de este articulo es resolver un sistema de ecuaciones de la forma $AX = Y$, con $A = (a_{ij}) \in M_{m\times n}(S)$, $Y \in S^m$, S siendo el semianillo $\min - \max$. Si el sistema tiene solución, podemos calcular su solución máximal respecto del orden natural de S. Además, presentamos el criptoanalisis de un protocolo de intercambio de clave sobre matrices y polinomios con entradas en dicho semianillo, junto con el coste computacional del mismo.

1 Introducción

Un semianillo $(S, +, \cdot)$ es un conjunto S con dos operaciones internas, $+$ y $\cdot$, donde $(S, +)$ es un monoide conmutativo y $(S, \cdot)$ es un monoide, estando ambas operaciones internas conectadas por una distributividad similar a la de un anillo.

El algebra lineal sobre el semianillo $(S, \oplus, \otimes) = (\mathbb{R}, \min, \max)$ tiene un gran interes, por ejemplo en [1] se estudian los sistemas de ecuaciones obre este semianillo, y en [2] el calculo de autovalores y autovectores, resultando en aplicaciones prácticas directas como por ejemplo en [3] en el que dichos sistemas se utilizan para la sincronización de semáforos.

En este artículo, mostramos un nuevo metodo para resolver sistemas de ecuaciones sobre el semianillo tropical $\min - \max$ de la forma $AX = Y$ con $A = (a_{ij}) \in M_{m\times n}(S)$, $Y \in S^m$, además de mencionar una posible aplicación en criptografia.

2 Resultados

Empezaremos recordarndo algunos conceptos importantes necesarios para el desarrollo del trabajo. Para ello, notar que sobre los vectores con entradas en el semianillo $\min - \max$ podemos introducir un orden, dado por

$$X = (x_1, \dots x_n) \geq Y = (y_1, \dots, y_n) \text{ si y solo si } x_i \geq y_i \ \forall i = 1, \dots, n. \tag{1}$$

para todo $X, Y \in S^n$. Además, se puede demostrar que dicho orden en compatible con la multiplicacion de matrices $Mat(S)_{n\times n}$.

Sea $AX = Y$ un sistema de ecuaciones en R con indeterminadas $x_1, \dots, x_n$,

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(m-1)1} \\ a_{m1} \end{pmatrix} \otimes x_1 \oplus \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{(m-1)2} \\ a_{m2} \end{pmatrix} \otimes x_2 \oplus \cdots \oplus \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{(m-1)n} \\ a_{mn} \end{pmatrix} \otimes x_n = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{pmatrix}, \tag{2}$$

con $a_{i,j}, y_j \in S$ para todo $i = 1, \ldots, n$ $j = 1, \ldots, m$. Sea A_j la $j-th$ columna de A, $A_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$, entonces, el sistema anterior puede ser escrito como

$$A_1 \otimes x_1 \oplus A_2 \otimes x_2 \oplus \cdots \oplus A_n \otimes x_n = Y. \tag{3}$$

Definición 2.1. *Sea $(S, \oplus, \otimes)$ el semianillo $\min - \max$, y sea $AX = Y$ un sistema lineal de ecuaciones. Decimos que $\hat{X}$ es la solución máxima del sistema si y solo si se satisfacen las dos condiciones siguientes:*

1. *$\hat{X} \in R^n$ es una solución del sistema, es decir, $A\hat{X} = Y$,*
2. *si $Z \in R^n$ es cualquier otra solución del sistema, entonces $Z \leq \hat{X}$.*

El siguiente resultado proporciona una formula explicita para el cálculo de la solución máximal de un sistema de ecuaciones sobre este anillo.

Teorema 2.2. *Sea $(S, \oplus, \otimes)$ el semianillo $\min - \max$, sea $W_i = \{x \in S : x \otimes A_i \oplus Y = Y\} \forall i = 1, \ldots, n$. Supongamos que estos subconjuntos tienen un máximo con respecto al orden inducido en S:*

$$C_i = \max W_i.$$

Si $XA = Y$ tiene solución, entonces $\hat{X} = (C_1, \ldots, C_n)$ es la solución máxima del sistema.

Proof. Si existe una solución $Z = (z_1, \ldots, z_n)$, entonces, basta con probar que $z_k \otimes A_k \leq Y$ para todo $k = 1, \ldots, n$, y por lo tanto, podemos demostrar que $z_k \in W_k$. Como consecuencia, $\hat{X} \geq Z$. Finalmente, demostramos que $\hat{X}$ es una solución, y por lo tanto, es la solución máxima. □

Otro ejemplo de aplicación de este semianillo, aparece en [5], donde se propone un protocolo de intercambio de clave con el uso de matrices sobre el semianillo $\min - \max$. Dicho protocolo es un caso particular del propuesto por [6], donde se cambian los semianillos simples finitos por el semianillo $\min - \max$.

Sea $A, B \in Mat(S)_{n \times n}$. Los polinomios de grado m con entradas en $\mathbb{Z}$ lo denotaremos $\mathbb{Z}_m[x]$. Sea $J = \mathbb{Z}_m[x] \times \mathbb{Z}_m[x]$. Se utilizará la acción $\phi : J \times Mat_n(S) \longrightarrow Mat_n(S)$ dada por $(p(x), q(x)) \times M \longrightarrow p(A) \cdot M \cdot q(B)$.

1. Alice y Bob toman n, m números naturales, $M_1, M_2, S \in Mat_n[R]$ y los hacen públicos.
2. Alice elige polinomios $p_a, q_a \in \mathbb{Z}_m[x]$ que mantendrá en privado. Alice computa $A = \phi((p_a, q_a), S) = p_a(M_1)Sq_a(M_2)$. Alice envía A a Bob.

3. Bob elige polinomios $p_b, q_b \in \mathbb{Z}_m[x]$ que mantendrá en privado. Bob computa $B = \phi((p_b, q_b), S) = p_b(M_1)Sq_b(M_2)$. Bob envía B a Alice.
4. Su clave secreta común es

$$\phi((p_a, q_a), B) = \phi((p_a p_b, q_a q_b), S) = \phi((p_b, q_b), A)$$

$$p_a(M_1)Bq_a(M_2) = p_a(M_1)p_b(M_1)Sq_b(M_2)q_a(M_2) = p_b(M_1)Aq_b(M_2)$$

Teorema 2.3. *Sea R un semianillo finito aditivamente idempotente, y sean las matrices $M_1, S, M_2, A \in Mat_n(R)$. Supongamos que existen dos polinomios $\varphi, \psi \in C_m[x]$ tales que $\varphi(M_1)S\psi(M_2) = A$. Sea W el conjunto*

$$W = \big\{(x, a, b) \in C \times \{0, 1, \ldots, m\} \times \{0, 1, \ldots, m\} : xM_1^a S M_2^b + A = A\big\}.$$

Entonces

$$F_S[X, Y] = \sum_{(k,i,j)\in W} kX^i SY^j \tag{4}$$

verifica que

$$F_S[M_1, M_2] = A \tag{5}$$

Por el cual, podemos obtener el siguiente resultado

Teorema 2.4. *Sea $n \times n$ el tamaño de las matrices usadas en el protocolo de intercambio de claves. Si suponemos m una cota superior de los grados de los polinomios privados, entonces el existe un algoritmo capaz de recuperar la clave común a partir de la información pública con un coste de operaciones $O(m^2 n^3)$.*

En el artículo [7] se puede obtener un criptoanalisis del mismo en el caso de semianillos simples finitos. Gracias a los resultados anteriores, podemos obtener el siguiente teorema

3 Conclusiones

En este trabajo se ha proporcionado un algoritmo explicito con formula explicita para el cálculo de la solución maximal de un sistema de ecuaciones lineales sobre el anillo $\min - \max$. El uso de esta fórmula nos permite un criptoanalisis similar al realizado en [7] sobre otro intercambio de clave propuesto en [6] para la obtención de la clave compartida por dos partes a partir unicamente de la informaicón pública intercambiada y que hace uso de matrices sobre el semianillo $\min - \max$ tal cual se propone en [5], probando de este modo la inseguridad de dicho esquema.

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λ-curvatura de Forman-Ricci en grafos*

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Resumen. Se han propuesto varias versiones discretas de la curvatura de Ricci, ya que las propiedades geométricas de una red se utilizan para comprender información importante asociada a ella. En este artículo obtenemos algunas propiedades importantes de la λ-curvatura de Forman-Ricci, un concepto que generaliza e integra la curvatura de Forman-Ricci y la curvatura de Forman-Ricci aumentada. Mostramos que esta definición captura la esencia de la curvatura de Ricci en variedades riemannianas, al probar análogos discretos de un resultado importante en geometría riemannianas: el teorema de Bonnet-Myers. Además, estudiamos la relación entre la λ-curvatura de Forman-Ricci y la hiperbolicidad de Gromov, una propiedad métrica que extiende a los grafos el concepto de curvatura negativa.

1. Introducción

La hiperbolicidad de Gromov en grafos es una versión discreta del concepto de curvatura negativa. Ha sido fundamental para comprender las propiedades estructurales de una red, ya que esta conceptualización está relacionada con la noción de la columna vertebral de la red y sus vías de comunicación (véase [7,13,16]). Por lo tanto, el estudio de la hiperbolicidad de Gromov en grafos se ha convertido en un tema de creciente interés (véase, por ejemplo, [2, 13, 14, 21]).

La curvatura de Ricci es una herramienta básica en la geometría de Riemann que caracteriza las propiedades geométricas locales de las variedades de Riemann al relacionar la tasa local de crecimiento del volumen con la dispersión geodésica. Como la curvatura de Ricci encapsula información esencial en geometría y campos relacionados (por ejemplo, las ecuaciones de campo de Einstein en la relatividad general), ha habido múltiples intentos de extender de manera provechosa su definición a otros dominios. Por lo tanto, se han propuesto varias versiones discretas de la curvatura de Ricci (ver [3, 5, 8, 9]), ya que las propiedades geométricas de una red se utilizan para comprender información importante asociada con ella.

Las dos versiones iniciales de la curvatura de Ricci discreta para grafos se deben a Ollivier (ver [8, 9]) y Forman (ver [5]). La curvatura de Ollivier-Ricci se basa en la teoría del transporte óptimo: la idea clave en su definición es utilizar la distancia de Wasserstein. Aporta información

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sobre las propiedades estructurales de las redes, como las propiedades de expansión y las estructuras de la comunidad. La curvatura de Forman-Ricci es otro análogo discreto de la curvatura de Ricci, pero se basa en un enfoque combinatorio, utilizando formas diferenciales discretas: se inspira en el método de Bochner, que en el caso clásico permite escribir el operador laplaciano como la suma de un término similar al laplaciano y un término que depende de la curvatura de Ricci. La curvatura de Forman-Ricci es útil para estudiar propiedades relacionadas con la estructura topológica y geométrica del gráfico.

La curvatura de Forman-Ricci aumentada es una extensión del concepto de curvatura de Forman-Ricci. Se han estudiado variantes de este concepto en [19]. La curvatura de Forman-Ricci aumentada muestra las relaciones entre la estructura de la comunidad y los grupos de curvaturas en una red (véase [1, 12, 17, 18, 22]). Estas curvaturas proporcionan una herramienta para estudiar los efectos dinámicos, resultantes del flujo de información, en redes complejas. Las regiones de curvatura (alta o baja) describen el comportamiento de crecimiento o contracción de la red (véase [11, 20]).

Para definir estas curvaturas, siempre consideramos grafos que son conexos y localmente finitos (es decir, cada vértice tiene un número finito de vecinos). Como es usual, el conjunto de vértices de un grafo G se denota por $V(G)$, y el conjunto de aristas se denota por $E(G)$. Si $u \in V(G)$, denotamos por $N(u)$ el conjunto de vecinos de u, es decir, $N(u) = \{v \in V(G) : uv \in E(G)\}$ y definimos el grado $\deg(u)$ del vértice u, como la cardinalidad de $N(u)$.

La *curvatura de Forman-Ricci* y la *curvatura de Forman-Ricci aumentada* de una arista uv del grafo G se definen, respectivamente, como

$$F(uv) := 4 - \deg(u) - \deg(v),$$

$$F_3(uv) := F(uv) + 3 \cdot \sharp\{\text{triángulos} \ni uv\},$$

donde $\sharp\{\text{triángulos} \ni uv\}$ denota el número de triángulos (3-ciclos) en los que $uv \in E(G)$ está contenido (véase [4, 5, 10, 11]).

En este trabajo estudiamos algunas propiedades importantes de la *λ-curvatura de Forman-Ricci*. Este concepto permite estudiar de forma unificada tanto la curvatura de Forman-Ricci como la curvatura de Forman-Ricci aumentada, cuando $\lambda = 0$ y $\lambda = 3$, respectivamente. Para una $\lambda \in \mathbb{R}$ fija, la *λ-curvatura Forman-Ricci* de una arista uv del grafo G se define como

$$F_\lambda(uv) := F(uv) + \lambda \cdot \sharp\{\text{triangles} \ni uv\}.$$

Una característica interesante de la definición de esta curvatura discreta es que es local, como la curvatura de Ricci en geometría riemanniana.

Mostramos que esta definición captura la esencia de la curvatura de Ricci en las variedades de Riemann, al probar análogos discretos de un resultado importante en la geometría de Riemann, como el Teorema de Bonnet-Myers (ver Teoremas 2.3 y 2.4).

También estudiamos la relación entre la curvatura λ-Forman-Ricci y la hiperbolicidad de Gromov (ver Teoremas 2.1 y 2.2).

2. Resultados

Decimos que un grafo es *t-hiperbólico*, con $t \geq 0$, si cualquier lado en cada triángulo geodésico está contenido en el t-entorno de la unión de los otros dos lados. Definimos la *constante de hiperbolicidad* $\delta^*(G)$ del grafo G como el ínfimo de las constantes $t \geq 0$ tales que G es t-hiperbólico. Decimos que G es *hiperbólico* (o *hiperbólico de Gromov*) si es t-hiperbólico para algún $t \geq 0$. La hiperbolicidad en grafos es una versión discreta del concepto de curvatura negativa.

Los siguientes resultados relacionan la curvatura λ-Forman-Ricci y la constante de hiperbolicidad.

Teorema 2.1. *Si $F_\lambda \geq k$ en un grafo G para algunas constantes reales que satisfacen $\lambda \leq 0$ y $k > 0$ ó $0 \leq \lambda < 2$ y $k > \lambda$, entonces $\delta^*(G) = 0$.*

Teorema 2.2. *Sea G un grafo y $\lambda < 2$ con $F_\lambda \geq 0$. Si G tiene algún triángulo, entonces su constante de hiperbolicidad es como máximo $3/2$.*

Un resultado importante en la geometría riemanniana es el teorema de Bonnet-Myers: si (M, g) es una variedad riemanniana completamente conexa con curvatura de Ricci acotada inferiormente por $1/r^2 > 0$, entonces M es una variedad compacta con diámetro como máximo πr. (En particular, esto implica que una variedad con un límite inferior positivo de su curvatura de Ricci tiene una constante de hiperbolicidad finita). Tenemos resultados similares para F_λ.

Teorema 2.3. *Supóngase que $F_\lambda \geq k$ en un grafo conexo G para algunas constantes reales que satisfacen $\lambda \leq 0$ y $k > 0$ o $0 \leq \lambda < 2$ y $k > \lambda$. Entonces G es un grafo con un diámetro de como máximo 2.*

Teorema 2.4. *Sea G un grafo y $\lambda < 2$ con $F_\lambda \geq 0$. Si G tiene algún triángulo, entonces G es un grafo con un diámetro de como máximo 4.*

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Exploring Biological Processes in the framework of Boolean Networks

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Abstract. A straightforward method is explored to study biological processes, such as tumor evolution, by inferring a gene regulatory network that models the process. To take advantage of the available biological information, the proposed framework is that of Boolean networks. In this context, cell types correspond to stable configurations of the dynamical system, known as attractors. The goal is to construct a Boolean network with a higher number of attractors, representing potential stable states in the process's evolution. Using the resulting Boolean network, a theoretical trajectory of the biological process can be described and compared with experimental data. Boolean modeling serves as a discrete approximation of the system of differential equations that describe genetic network transitions, which become computationally intractable as the number of genes increases. This approach is exemplified with a process in cancer.

1 Introduction

Biological processes are intricate systems that can be modeled as interconnected networks across various spatial and temporal scales, from genetic networks to organ networks within an organism. This Systems Biology approach aims to integrate these levels, starting from genes that activate or inhibit through interactions. Genetic networks, modeled as dynamical systems [1], evolve toward attractors, which correspond to stable cellular states.

In this sense, Boolean network models have been widely used to explore the evolution of many processes, including diseases such as cancer [2], [3], [4]. Here network nodes are generally genes, which can be activated or inhibited, and edges reflect their interaction. The variation in the state of a gene from one time t to the next is modelled as a Boolean function of the set of relevant genes at that transition. The system of all functions is a dynamic system whose attractors would be stable biological states, that is, cell types [5]. From the constructed Boolean network, it is possible to describe a theoretical evolution of the biological process that can be checked with experimental data. Then a Boolean network suggests cellular states that can be tested by experimentation.

We start by setting up a graph that reflects the known interactions of the process to be studied, then inferring the Boolean functions, and analyzing the evolution of the model, paying particular attention to attractors. The main difficulty is deciding the Boolean functions, given

that the set of all possible functions can be very high. It is necessary to look for ways to restrict the number of possibilities, and one way that has been used effectively is to resort to the concept of canalization [6], [7]. More specifically, focusing on the so-called nested canalizing functions [8], [9]. Even so, we still find a high number of functions. Then, other possible restrictions are to impose on the system attractors that the biology of the process manifests [9]. That is, from biology, we can establish a set of attractors that should be present in all the networks built. In this way, we restrict the total number of possible networks to those compatible with these attractors.

Our interest is to use this approach in the exploration of tumour processes. A tumour cell can be considered as an attractor [4] so that from our data we will have information to decide on some attractors of our dynamic model. In this work we test this idea on a cancer process.

2 Exploring biological networks. Methodology

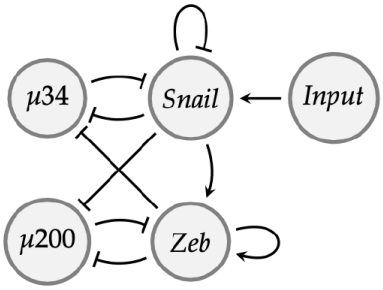
To explore a biological process in the above framework, the first thing to do is collect the available information about that process and perhaps design experiments, to know how genes interact and detect states that we can consider attractors of a dynamical system. The immediate objective will be to build a graph whose nodes are genes, which synthesizes biological information, to study the evolution of the network from an initial configuration of the state of genes. Therefore we may consider some attractors that the system must necessarily have.

The next step would be to design an algorithm that builds all possible networks from the graph compatible with those attractors. The number of networks can be very high, so the algorithm must contain enough filters to eliminate networks. The general idea would be to confront the networks with the biology of the process. The structural and dynamic constraints proposed in [9] to decrease the number of networks are biologically motivated. For this purpose, nested canalizing and sign-compatible Boolean functions are considered, moreover ensuring that all networks include the biologically identified attractors. With this, it is expected to be able to find new attractors, which would be possible states of the process to be investigated. This in turn provides clues for experimental exploration of the process.

3 Results

Using the Epithelial-Mesenchymal Transition as an example, it can be represented in the graph shown in Figure 1. The reasonable network obtained using the methodology is also shown in Figure 1.

Starting with two initial attractors, one epithelial and one mesenchymal atractor, we found a third attractor that could be considered an intermediate state between the epithelial and the mesenchymal. Indeed various mathematical models have already introduced hybrid states in



$$Input = Input$$
$$Snail = Input \wedge (\neg Snail \vee \neg \mu 34)$$
$$\mu 34 = \neg Snail \wedge \neg Zeb$$
$$Zeb = Snail \vee (Zeb \wedge \neg \mu 200)$$
$$\mu 200 = \neg Snail \wedge \neg Zeb$$

Figure 1: Epithelial-Mesenchymal Transition modeling: Graph representing EMT system and elected Boolean Network from the networks obtained by the procedure

EMT [10], and currently the EMT process is seen as a set of multiple intermediate states between the epithelial and mesenchymal phenotypes [11].

4 Conclusions

We wanted to illustrate a fairly straightforward approach to exploring biological processes, which nonetheless could be very effective. We think that it would be a good way to start exploring tumour evolution processes, in order to build a model based on Boolean Networks. In principle, it would be easy to select states of the process established in the current knowledge of the disease under study and thus establish some attractors from the outset. Then, from there, the idea is to look for other meaningful states that allow a better understanding of the evolution of the process.

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Domination, matching and transversal numbers for Berge-G hypergraphs*

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Abstract. Let $G = (V(G), E(G))$ be a graph and $H = (V(H), E(H))$ be a hypergraph. The hypergraph H is a *Berge-G* if there is a bijection $f : E(G) \mapsto E(H)$ such that for each $e \in E(G)$ we have $e \subseteq f(e)$. We define *dilations of* G as a particular subfamily of not necessarily uniform Berge-G hypergraphs. We examine domination, matching and transversal numbers and some relation between these parameters in that family of hypergraphs. Our work generalizes previous results concerning generalized power hypergraphs.

Keywords: Berge-G, dilation, domination, matching, transversal, extremal.

1 Preliminaries

A *(finite) hypergraph* $\mathcal{H} = (V(H), E(H))$ consists of a (finite) set $V(H)$ and a collection $E(H)$ of non-empty subsets of $V(H)$. The elements of $V(H)$ are called *vertices* and the elements of $E(H)$ are called *hyperedges*, or simply *edges* of the hypergraph. A k*-uniform* hypergraph is a hypergraph such that each edge consists of k vertices. The *rank* of H is the maximum size of its hyperegdes.

Two vertices of $\mathcal{H}$, u and v, are *adjacent* if there is an edge e such that $u, v \in e$. The vertex v is *incident* to the edge e if $v \in e$. The *degree* of a vertex v is the number of edges incident to v. All over the paper, a *graph* will be a 2-uniform hypergraph.

Let $G = (V(G), E(G))$ be a graph and $H = (V(H), E(H))$ be a hypergraph. According to [5], the hypergraph H is a *Berge-G* if there is a bijection $f : E(G) \mapsto E(H)$ such that for each $e \in E(G)$ we have $e \subset f(e)$. In other words, given a graph G a Berge-G can be constructed by replacing each edge of G with a hyperedge that contains it (been allowed to introduce new vertices). G is called *the support graph of* H and its vertices are named *support vertices of* H. Observe that a family of hypergraphs is obtained from the graph G and it is denoted $\mathcal{B}(G)$.

As a particular case of Berge-G hypergraph, for any $k \geq 3$ and $1 \leq s \leq \frac{k}{2}$, the *generalized power* of G, denoted by $G^{k,s}$, is defined ([7]) as the k-uniform hypergraph with vertex set $V(G^{k,s}) = \bigcup\{\mathbf{v} : v \in V\} \bigcup \{\mathbf{e} : e \in E\}$, and edge set $E(G^{k,s}) = \{\mathbf{u} \cup \mathbf{v} \cup \mathbf{e} : e = \{u, v\} \in E\}$, where $\mathbf{v}$ is an s-set containing v and $\mathbf{e}$ is a $(k - 2s)$-set corresponding to e. Each vertex in

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$\mathbf{v} - v$ is said to be *a copy of* v. For $s = 1$, $G^{k,1} = G^k$ is the *kth*-power hypergraph of G and there is no copy of v in G^k. Intuitively, $G^{k,s}$ is obtained from G by replacing each vertex v by an s-subset $\mathbf{v}$ and each edge by a k-set obtained from $\mathbf{v} \cup \mathbf{u}$ by adding $(k - 2s)$ new vertices. It is readily checked that $G^{k,s} \in \mathcal{B}(G)$ for any $k \geq 3$ and $1 \leq s \leq \frac{k}{2}$.

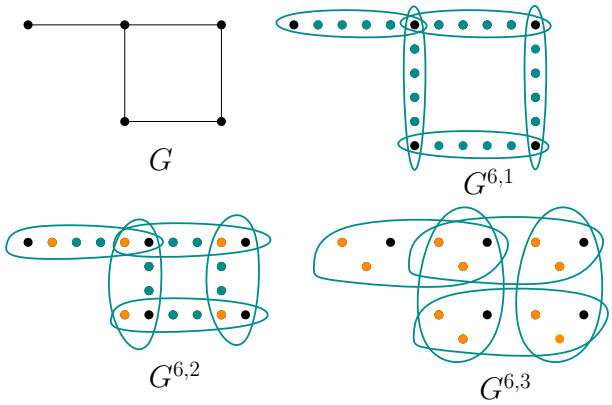

Figure 1: The graph G, the 6th-power hypergraph G^6, the generalized power hypergraphs $G^{6,2}$ and $G^{6,3}$.

Definition 1.1. *Let $G = (V(G), E(G))$ be a graph and $k \geq 3$. For each pairwise adjacent vertices v_i, $v_j \in V(G)$, let s_i and s_j be two positive integers related to v_i and v_j, respectively, and such that $2 \leq s_i + s_j \leq k$. A* dilation *of G is a hypergraph $H = (V(H), E(H))$ of rank k whose vertex set is*

$$V(H) = \bigcup \{\mathbf{v_i} : v_i \in V(G)\} \cup \{\mathbf{e} : e \in E(G)\}$$

and whose hyperedges set is

$$E(H) = \{\mathbf{v_i} \cup \mathbf{v_j} \cup \mathbf{e} : e = \{v_i, v_j\} \in E(G)\},$$

where $\mathbf{v_k}$ is a s_k-set containing v_k ($k \in \{i, j\}$) and $\mathbf{e}$ is a set of size not greater than $k - s_i - s_j$ which corresponds to the edge e. A vertex in $\mathbf{v_k} - v_k$ is called copy vertex *and a vertex in $\mathbf{e}$ is called* additional vertex.

The family of all dilations of a graph G is denoted $\Gamma(G)$, and it is readily noticed that $G^{k,s} \in \Gamma(G) \subsetneq \mathcal{B}(G)$, for any $k \geq 3$ and $1 \leq s \leq \frac{k}{2}$ and $G \neq K_2$.

$\Gamma_0(G)$ denotes the set of dilations without additional vertices in their hyperedges. $\Gamma_1(G)$ denotes the set of dilations with at least one additional vertex in every hyperedge.

Clearly, for any graph G, we get $\Gamma_0(G) \cap \Gamma_1(G) = \emptyset$, and for $G \neq K_2$, it is verified $\Gamma_0(G) \cup \Gamma_1(G) \subsetneq \Gamma(G)$, $G^{k,\frac{k}{2}} \in \Gamma_0(G)$ for any $k \geq 3$ and $G^{k,s} \in \Gamma_1(G)$ for any $k \geq 3$ and $1 \leq s < \frac{k}{2}$.

Next, let us recall the parameters of our study. Given a hypergraph $H = (V, E)$, $D \subset V$ is a *dominating set* of H if for every $v \in V - D$ there exists $u \in D$ such that u and v are adjacent. The minimum cardinality of a dominating set of $\mathcal{H}$, $\gamma(H)$, is its *dominating number*. A *matching* in H is a set of disjoint hyperedges. The *matching number* of H, $\nu(H)$, is the

maximum size of a matching in H. A subset $T \subset V$ is a *transversal* (or a *vertex cover*) of H if T has nonempty intersection with every hyperedge of H. The *transversal number* of H, $\tau(H)$, is the minimum size of a transversal of H.

2 Results

For any graph with no vertex of degree 0 G, it is known that $\gamma(G) \leq \nu(G) \leq \tau(G)$. For any hypergraph H, it is clear that $\gamma(H) \leq \tau(H)$ and $\nu(H) \leq \tau(H)$ hold. However, the inequality $\gamma(H) \leq \nu(H)$ does not hold, for the general case.

To find out hypergraphs where the corresponding parameters equalities are reached in the whole set of hypergraphs is a NP-hard problem ([1]). This fact leads us to look for characterizations in a more restrictive family of hypergraphs.

As a starting point, we study properties of domination, matching and transversal numbers of G that are inherited by the Berge-G hypergraphs. We get the following

Theorem 2.1. *Let $G = (V, E)$ be a simple graph. Then:*

1. $\nu(H) = \nu(G)$, $\tau(H) = \tau(G)$ *and* $\gamma(G) \leq \gamma(H) \leq \tau(G)$, *for all* $H \in \Gamma(G)$.
2. $\gamma(H) = \gamma(G)$, *for all* $H \in \Gamma_0(G)$.
3. $\gamma(H) = \tau(G)$, *for all* $H \in \Gamma_1(G)$.

We extend some results related to connected generalized power hypergraphs proven in [4] and present them within the context and terminology of dilations hypergraphs. In fact, this improving is quite direct since proofs also work when the condition of uniformity on generalized power hypergraphs is replaced by the conditon of *each hyperedge contains at least one additional vertex*. Therefore, we can state the following

Theorem 2.2. *Let G be a graph. Then:*

1. $\nu(H) \leq \gamma(H) \leq 2\nu(H)$, *for all* $H \in \Gamma_1(G)$.
2. $\gamma(H) \leq \nu(H)$, *for all* $H \in \Gamma_0(G)$.

In this context, the hypergraph H is said to be *extremal for the domination number* if it verifies one of the equalities $\gamma(H) = \nu(H)$ or $\gamma(H) = 2\nu(H)$. The characterization of such extremal generalized power hypergraphs for the domination number is already introduced in [4], however it is incomplete. We complete and extend that characterization for the family of dilations and we get the following

Theorem 2.3. *For $H \in \Gamma_1(G)$,*

1. $\gamma(H) = 2\nu(H)$ *if and only if* $G = K_{2\nu(H)+1}$.
2. $\gamma(H) = \nu(H)$ *if and only if* $\tau(G) = \nu(G)$.

A graph G satisfies the König-Egerváry property if $\tau(G) = \nu(G)$ and it is called König-Egerváry graph (KEG for short). These kind of graphs are widely studied ([2]). Observe that bipartite graphs are KEG but they are not the only ones.

For the extremal hypergraphs for the domination number in the dilations set $\Gamma_0(G)$ several families of support graphs must be considered. Namely: the families $\mathcal{G}^B_{\geq 2}$ and $\mathcal{G}^{NB}_{\geq 2}$ described in [8] and the family $\mathcal{G}_1$ described in [6], although they use different notations.

Theorem 2.4. *For $H \in \Gamma_0(G)$, $\gamma(H) = \nu(H)$ is and only if $G \in \mathcal{G}^B_{\geq 2} \cup \mathcal{G}^{NB}_{\geq 2} \cup \mathcal{G}_1$.*

Finally, we present two results ensuring the existence of dilations for each non extremal values of $\gamma(H)$ for $H \in \Gamma_0(G) \cup \Gamma_1(G)$.

Theorem 2.5. *Let $n \geq 2$ be an integer number. For any $m \in \{n+1, \ldots, 2n-1\}$ there is a graph G such that $\nu(H) = n$ and $\gamma(H) = m$, for all $H \in \Gamma_1(G)$.*

Theorem 2.6. *Let $n \geq 2$ be an integer number. For any $m \in \{1, \ldots, n-1\}$ there is a graph G such that $\nu(H) = n$ y $\gamma(H) = m$, for all $H \in \Gamma_0(G)$.*

3 Problems

1. Finding an *iff* condition for a hypergraph to be Berge-G of a simple G.
2. Characterizing, by means of the support graph G, the families of dilations for which each non-extremal value of $\gamma(H)$ given by Theorems 2.5 and 2.6 are reached.

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Variety of mutual-visibility problems on graphs of diameter two

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Abstract. The mutual-visibility problem in graphs regards finding a largest set of vertices S of the graph satisfying that any two vertices $x, y \in S$ are visible with respect to S, namely, there is a shortest x, y-path P whose all internal vertices of P are not in S. Variations of this problem are known, based on the extension of the visibility property to those vertices that are included or not in S. These variations are called total, outer and dual mutual-visibility problems. The four visibility parameters, regarding such problems, are considered in this work for the special case of graphs of diameter two.

1 Introduction

Given a connected graph G and a set of vertices $X \subseteq V(G)$, two vertices $x, y \in V(G)$ are called X-*visible* if there is a shortest x, y-path whose interior vertices are not in X. Let $X \subseteq V(G)$. Then the following concepts are known from [1].

• *Mutual-visibility set*: if any two vertices of X are X-visible.

• *Outer mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x \in X$ and $y \in \overline{X}$ are X-visible.

• *Dual mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x, y \in \overline{X}$ are X-visible.

• *Total mutual-visibility set*: if any two vertices $x, y \in V(G)$ are X-visible.

Regarding such graph structures, the parameters mutual-visibility number, outer mutual-visibility number, dual mutual-visibility number, and total mutual-visibility number are defined as the cardinalities of the largest (respectively) mutual-visibility sets from the above ones. The corresponding parameters are denoted by $\mu(G)$, $\mu_o(G)$, $\mu_d(G)$, and $\mu_t(G)$. Several contributions on these parameters are already known.

In this work we consider these four parameters while studying some graphs of diameter two. Moreover, the results of this work have been already published in [3].

2 Results

In [2] it was shown that for the integers $m, n \geq 2$, it holds $\mu(K_m \Box K_n) = z(m, n; 2, 2)$, where $z(m, n; 2, 2)$ is the maximum number of 1s that an $m \times n$ binary matrix can have, provided that it contains no 2×2 submatrix of 1s, also known as (an instance of) the Zarankiewicz's problem. Also, for the total mutual-visibility number it is known from [4] that $\mu_t(K_n \Box K_m) =$

$\max\{n, m\}$. for $n, m \geq 2$. For the dual and the outer mutual-visibility number, we have the following related respective results.

Theorem 2.1. *If $n, m \geq 3$, then*

- $\mu_d(K_n \Box K_m) = n + m - 1$,
- $\mu_o(K_n \Box K_m) = n + m - 2$.

For the case of the direct product of complete graphs we prove the following result.

Theorem 2.2. *If $n, m \geq 5$, then $\mu_t(K_n \times K_m) = \mu(K_n \times K_m) = nm - 4$.*

We next consider the line graphs of complete graphs and of complete bipartite graphs. First, note that if $n \geq 4$, then $diam(L(K_n)) = 2$, and if $m, n \geq 2$, then $diam(L(K_{m,n})) = 2$. In general, if $diam(G) \leq 2$, then $diam(L(G)) \leq 3$, and we begin with a characterization of mutual-visibility sets in line graphs $L(G)$ for graphs G with $diam(G) = 2$.

Lemma 2.3. *Let G be a graph of diameter 2 and $F \subseteq E(G)$. Then $S_F \subseteq V(L(G))$ is a mutual-visibility set of $L(G)$ if and only if for any two independent edges $uv, u'v' \in F$ one of the following conditions is satisfied.*

- *There is an edge $xy \notin F$ incident with both uv and $u'v'$, or*
- *$d_{L(G)}(e_{uv}, e_{u'v'}) = 3$ and there is a vertex $z \in V(G)$ adjacent to (without loss of generality) u and u' in G, such that $uz, u'z \notin F$.*

The result above is useful to obtain the following contributions.

Theorem 2.4. *Let $n \geq 3$ be an integer and $F \subseteq E(K_n)$. Then $S_F \subseteq V(L(K_n))$ is a μ-set of $L(K_n)$ if and only if $(K_n)_F \cong T(n, 3)$.*

Corollary 2.5. *If $n \geq 3$, then $\mu(L(K_n)) = (\frac{2}{3} + o(1))\,\frac{n^2}{2}$.*

With respect to the other mutual-visibility parameters of the graph $L(K_n)$, the following observations can be used. If F is a set of edges of K_n, then the corresponding set S_F in $L(K_n)$ has the (total, outer or dual) mutual-visibility properties based on the existence of certain structures obtained from pairs of not incident edges from $E(K_n)$, F, or $E(K_n) \setminus F$. Recall that a pair of not incident edges $uv, u'v' \in E(K_n)$ are S_F-visible in $L(K_n)$ whenever there is an edge $xy \notin F$ such that (without loss of generality) $x = u$ and $y = u'$. These facts, the definitions of (total, outer or dual) mutual-visibility sets and the structure of $L(K_n)$ allow to readily observe the following result, whose proof is rather simple and left to the reader.

Lemma 2.6. *Let $n \geq 3$ be an integer and let $F \subseteq E(K_n)$. Then,*

- *S_F is a total mutual-visibility set of $L(K_n)$ if and only if for any two not incident edges $uv, u'v' \in E(K_n)$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$.*
- *S_F is a dual mutual-visibility set of $L(K_n)$ if and only if*

– for any two not incident edges $uv, u'v' \in E(K_n) \setminus F$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$, and

– for any two not incident edges $xy, x'y' \in F$ the subgraph induced by x, y, x', y' has at least one edge not in F different from xy and $x'y'$.

• *S_F is an outer mutual-visibility set of $L(K_n)$ if and only if*

– for any two not incident edges $uv, u'v'$ with $uv \in E(K_n) \setminus F$ and $u'v' \in F$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$, and

– for any two not incident edges $xy, x'y' \in F$ the subgraph induced by x, y, x', y' has at least one edge not in F different from xy and $x'y'$.

The lemma above is used to deduce the next results.

Proposition 2.7. *For any integer $n \geq 3$, $\mu_t(L(K_n)) \geq n - 1 + \lfloor \frac{n-1}{2} \rfloor$.*

The *Turán number* of a graph H, written $ex(n; H)$, is the maximum number of edges in an n vertex graph not containing H.

Theorem 2.8. *For any integer $n \geq 3$, $\mu_t(L(K_n)) = ex(n; C_4)$.*

Theorems 2.4 and 2.8 provide a way to compute $\mu(L(K_n))$ and $\mu_t(L(K_n))$. They are based on the analysis of forbidden subgraphs for $(K_n)_F$ where S_F is a mutual-visibility or a total mutual-visibility sets of $L(K_n))$, respectively. By using Lemma 2.6, an analysis of the *induced* forbidden subgraphs of $(K_n)_F$ for the (dual, outer, total) mutual-visibility set S_F of $L(K_n)$ shows that only three forbidden graphs are involved: K_4, K_4^-, and C_4. As proved in Theorem 2.4, K_4 is the only forbidden subgraph of $(K_n)_F$ for any mutual-visibility sets S_F of $L(K_n)$. As for the total mutual-visibility, all the three induced graphs are forbidden, but, since C_4 is a subgraph of both K_4 and K_4^-, it is sufficient to forbid only this graph, and then $\mu_t(L(K_n)) = ex(n, C_4)$, as stated by Theorem 2.8.

Similarly, for the outer mutual-visibility, the induced forbidden subgraphs are K_4 and K_4^-, and since K_4^- is a subgraph of K_4, we have the following result.

Theorem 2.9. *For any integer $n \geq 3$, $\mu_o(L(K_n)) = ex(n; K_4^-)$.*

The next known results is required in our deductions.

Theorem 2.10. [5, Theorem 1(a)] *If F has chromatic number k and a critical edge, and n is large enough, then $ex(n, F) = |E(T(n, k-1))|$. Moreover, $T(n, k-1)$ is the unique extremal graph.*

Since the graph K_4^- has chromatic number 3 and a critical edge, we deduce that the edges of K_n that form an outer mutual-visibility set of the largest cardinality in $L(K_n)$, together with the vertices in such edges, form a graph isomorphic to the Turán graph $T(n, 2)$. Recall that $T(n, 2)$ is the bipartite graph of order n with partite sets of cardinality $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. Thus, the following result holds.

Corollary 2.11. *For any large enough integer n, $\mu_o(L(K_n)) = \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$.*

For the dual mutual-visibility, the two induced forbidden subgraphs are K_4 and C_4. Then the following result holds.

Theorem 2.12. *Let $F \subseteq E(K_n)$. Then $S_F \subseteq V(L(K_n))$ is a dual mutual-visibility set of $L(K_n)$ if and only if $(K_n)_F$ is a (K_4, C_4)-free graph.*

We now consider the line graphs of a complete bipartite graph $K_{m,n}$, $m, n \geq 2$. It is well known that the line graph of a connected graph G is a nontrivial Cartesian product if and only if $G = K_{n,m}$, $n, m \geq 2$. So, $L(K_{m,n}) \cong K_m \Box K_n$. It is already known from [2] that $\mu(K_m \Box K_n) = z(m, n; 2, 2)$, where $z(m, n; 2, 2)$ is the Zarankiewicz number, that can also be seen as the maximum number of edges in a complete bipartite graph $K_{m,n}$ that has no 4-cycle. We now state that the same conclusion can be also obtained by using Lemma 2.3.

Theorem 2.13. *Let $n, m \geq 2$ and $F \subseteq E(K_{m,n})$. Then $S_F \subseteq V(L(K_{m,n}))$ is a μ-set of $L(K_{m,n})$ if and only if S_F is a largest set of vertices of $L(K_{m,n})$ such that $(K_{m,n})_F$ contains no 4-cycle.*

The following consequence of Theorem 2.13 follows from the above-mentioned observations from [2].

Corollary 2.14. *For any two integers $n, m \geq 2$, $\mu(L(K_{m,n})) = z(m, n; 2, 2)$.*

Finally, by again using the fact that the line graph of a complete bipartite graph $K_{n,m}$ is isomorphic to $K_n \Box K_m$, a result from [4], and Theorem 2.1 lead to the following consequence.

Corollary 2.15. *For any two integers $n, m \geq 2$, $\mu_t(L(K_{m,n})) = \max\{n, m\}$, $\mu_d(L(K_{m,n})) = n + m - 1$, and $\mu_o(L(K_{m,n})) = n + m - 2$.*

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Torneos k-existencialmente cerrados

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Resumen. Un torneo T es k-existencialmente cerrado ($k \geq 1$) si para todo subconjunto $A \subset V(T)$ tal que $|A| = k$ y para todo $B \subseteq A$, existe un vértice $x \notin A$ tal que x domina a todo elemento de B y todo elemento de $A - B$ domina a x.

En esta charla presentaremos una breve introducción del problema general y mostraremos ejemplos de familias infinitas de torneos k-existencialmente cerrados para $k = 2, 3$. Ver [1] .

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