Some results in fuzzy metric spaces

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Abstract

The problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. In particular, and by modifying a definition of fuzzy metric space given by Kramosil and Michalek, George and Veeramani have introduced and studied the following interesting notion of a fuzzy metric space: A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a set, $*$ is a continuous t-norm and M is a function defined on $X \times X \times]0, +\infty[$ with values in $]0, 1]$ satisfying certain axioms and M is called a fuzzy metric on X.
It is proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $B(x, \varepsilon, t) = \{x \in X, 0 < \varepsilon < 1, t > 0\}$ where $B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $\varepsilon \in]0, 1[$ and $t > 0$.
The topological space (X, τ) is said to be fuzzy metrizable if there is a fuzzy metric M on X such that $\tau = \tau_M$. Then, it was proved that a topological space is fuzzy metrizable if and only if it is metrizable. From then, several fuzzy notions which are analogous to the corresponding ones in metric spaces have been given. Nevertheless, the theory of fuzzy metric completion is, in this context, very different to the classical theory of metric completion: indeed, there exist fuzzy metric spaces which are not completable.
This class of fuzzy metrics can be easily included within fuzzy systems since the value given by them can be directly interpreted as a fuzzy certainty degree of nearness, and in particular, recently, they have been applied to colour image filtering, improving some filters when replacing classical metrics
In this lecture we survey some results and questions obtained in recent years about this class of fuzzy metric spaces.

From probabilistic metric spaces to fuzzy metric spaces

Probabilistic metric spaces where introduced by K. Menger [21] who generalized the theory of metric spaces. In the Menger's theory the concept of distance is considered to be statistical or probabilistic, i.e. he proposed to associate a distribution function $F_{x,y}$, with every pair of elements x, y instead of associating a number, and for any positive number t, interpreted $F_{xy}(t)$ as the probability that the distance from x to y be less than t.
Recall [28] that a distribution function F is a non-decreasing, left continuous mapping from the set of real numbers $\mathbb{R}$ into $[0, 1]$ so that $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

In the sequel H will denote the distribution function given by

$$H(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

We denote by Δ the set of distribution functions, and by Δ^+ the subset of Δ consisting of those distribution functions F such that $F(0) = 0$.

From probabilistic metric spaces to fuzzy metric spaces

A probabilistic metric space (briefly *PM* space) [28] is a pair $(X, \mathcal{F})$ such that X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ into Δ^+, whose value $\mathcal{F}(x, y)$ denoted by F_{xy}, satisfies for all $x, y, z \in X$:

(PM1) $F_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$.

(PM2) $F_{xy} = F_{yx}$

(PM3) If $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then $F_{xz}(t + s) = 1$.

Condition (PM1) is equivalent to the statement $x = y$ if and only if $F_{xy} = H$.
Conditions (PM1)-(PM3) are generalizations of the corresponding well-known conditions satisfied by a classical metric. In the beginning (PM3) was a controvert axiom (see [28]. Every metric space (X, d) may be regarded as a *PM* space. One has only to set $F_{xy}(t) = H(t - d(x, y))$ for each $x, y \in X$.
In his original formulation [21] Menger instead of (PM3) gave the following condition

(PM3)' $F_{xz}(t + s) \geq T(F_{xy}(t), F_{yz}(s))$ for all $x, y \in X,\ t, s \geq 0$

where T is a mapping from $[0, 1] \times [0, 1]$ into $[0, 1]$ satisfying

$T(c, d) \geq T(a, b)$ for $c \geq a,\ d \geq b$

$T(a, b) = T(b, a)$

$T(1, 1) = 1$

$T(a, 1) > 0$ for $a > 0$

It is immediately that (PM3)' contains (PM3).

From probabilistic metric spaces to fuzzy metric spaces

Nowadays, after the study in [28] of the axiomatic of probabilistic metric spaces, particularly, on the triangle inequality (PM3)', the next definitions are commonly assumed:
A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative

(ii) $a * 1 = a$ for every $a \in [0,1]$

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$

If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.
The three most commonly used continuous t-norms in fuzzy logic are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasievicz t-norm, denoted by $\mathfrak{L}$ ($x\mathfrak{L}y = \max\{0, x+y-1\}$). They satisfy the following inequalities:

$$x\mathfrak{L}y \leq x \cdot y \leq x \wedge y$$

and

$$x * y \leq x \wedge y$$

for each (continuous) t-norm $*$.

Definition

A Menger space is a triple $(X, \mathcal{F}, *)$ such that $(X, \mathcal{F})$ is a probabilistic metric space and $*$ is a t-norm such that for all $x, y, z \in X$ and $t, s \geq 0$:

(M4) $F_{xz}(t+s) \geq F_{xy}(t) * F_{yz}(s)$

From probabilistic metric spaces to fuzzy metric spaces

The following results were given in [28].

Definition

Let x be a point of the PM space $(X, \mathcal{F})$. The set of all points in X

$$B(x, r, t) = \{y \in X : F_{xy}(t) > 1 - r\}$$

where $r \in]0, 1[, \ t > 0$ is called a neighborhood of x.

According to this definition, a sequence $\{x_n\}$ in a *PM* space is said to converge to a point x (denoted $x_n \to x$) if for every $r \in]0, 1[, \ t > 0$ there exists $n_0 \in \mathbb{N}$ (n_0 depends on r and t) such that $x_n \in B(x, r, t)$ whenever $n \geq n_0$. Notice that $x_n \to x$ if, and only if, $F_{xx_n} \to H$, i.e. for every $t > 0$ $\lim_n F_{xx_n}(t) = 1$.
If $(X, \mathcal{F}, *)$ is a Menger space and $*$ is continuous then the family

$$\{B(x, r, t) : r \in]0, 1[, \ t > 0\}$$

is a base for a topology $\tau_{\mathcal{F}}$ on X, called topology induced by $\mathcal{F}$, and this topology is Hausdorff.
The family $\{B(x, r, t) : r \in]0, 1[, \ t > 0\}$ is a local base of each $x \in X$ in the topology $\tau_{\mathcal{F}}$.

From probabilistic metric spaces to fuzzy metric spaces

The completion of a Menger space was made by Sherwood in [29] (and later generalized in [30]) as follows.

Definition

Let $(X, \mathcal{F})$ *be a PM space. Then*

(a) A sequence of points $\{x_n\}$ *in* X *is a Cauchy sequence if* $F_{x_n x_m} \longrightarrow H$ *as* $n, m \to \infty$, *i.e.* $\lim_{n,m} F_{x_n x_m}(t) = 1$ *for all* $t > 0$.

(b) The space $(X, \mathcal{F})$ *is complete if every Cauchy sequence in* X *is convergent.*

(c) Let $(X, \mathcal{F})$ *and* $(Y, \mathcal{F}')$ *be two PM spaces. A mapping* $\varphi : X \to Y$ *is called an isometry if* $\mathcal{F}(x, y) = \mathcal{F}'(\varphi(x), \varphi(y))$ *for each* $x, y \in X$.

The *PM* spaces $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ are called isometric if there is a one-to-one isometry from X onto Y.

From probabilistic metric spaces to fuzzy metric spaces

Definition

Let $(X, \mathcal{F}, *)$ be a Menger space where $*$ is continuous. The Menger space $(X^*, \mathcal{F}^*, \diamond)$ is a completion of $(X, \mathcal{F}, *)$ if $(X, \mathcal{F})$ is isometric to a dense subset of $(X^*, \mathcal{F}^*)$ and $* = \diamond$.

In the set of all Cauchy sequences in X is defined the equivalent relation $\{x_n\} \equiv \{y_n\}$ if $\{F_{x_n y_n}\} \to H$, and the set of all equivalent classes determined by this relation is denoted by X^*.
Based on the completeness of the Levi metric space of distribution functions (Δ, L), it is possible to define

$$\mathcal{F}^*(p^*, q^*) = \lim_n{}^L \mathcal{F}(p_n, q_n)$$

for all $p^*, q^* \in X^*$, where $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences of p^* and q^*, respectively and it is proved that $(X^*, \mathcal{F}^*, *)$ is a complete Menger space.
Further, the mapping $\varphi : X \to X^*$, which assigns to each point x in X the equivalence class of Cauchy sequences determined by the constant sequence of value x, is an isometric embedding of X into X^* and $\varphi(X)$ is dense in X^*. This completion is unique up to an isometry.

From probabilistic metric spaces to fuzzy metric spaces

In 1965, Zadeh [33] introduced the concept of fuzzy set which transformed and stimulated almost all branches of Science and Engineering including Mathematics. A fuzzy set can be defined by assigning to each element of a set a value in $[0,1]$ representing its grade of membership in the fuzzy set. Mathematically, a fuzzy set A of X is a mapping $A : X \longrightarrow [0,1]$.
The concept of fuzziness found place in probabilistic metric spaces. The main reason behind this was that, in some cases, uncertainty in the distance between two points was due to fuzziness rather than randomness. With this idea, in 1975, Kramosil and Michalek [20] extended the concept of probabilistic metric spaces to the fuzzy situation as follows.

From probabilistic metric spaces to fuzzy metric spaces

Definition

[20, 6] The tern $(X, M, *)$ is a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times \mathbb{R}$ satisfying for all $x, y, z \in X,\ t, s \in \mathbb{R}$ the following axioms:

(KM1) $M(x, y, 0) = 0$ *for all* $t \leq 0$.

(KM2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$.

(KM3) $M(x, y, t) = M(y, x, t)$

(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, y, t + s)$

(KM5) The function $M_{xy} : \mathbb{R} \to [0, 1]$ *defined by* $M_{xy}(t) = M(x, y, t)$ *for all* $t \geq 0$ *is left continuous.*

(KM6) $\lim_{t \to \infty} M(x, y, t) = 1$

From the above axioms one can show that M_{xy} is an increasing function.
If $(X, M, *)$ is a fuzzy metric space we say that $(M, *)$ (or simply M) is a fuzzy metric on X.

From probabilistic metric spaces to fuzzy metric spaces

Any fuzzy metric M defined on X is equivalent to a Menger space (Corollary of Theorem 1) in the sense that for all $x, y \in X,\ t \in \mathbb{R}$

$$M(x, y, t) = F_{xy}(t)$$

Then, by the last formula, since $*$ is continuous, we can deduce from M a topology τ_M in an analogous way to that in Menger spaces. Moreover, if we translate the above concepts and results relative to completion in Menger spaces we obtain imitating the Sherwood's proof that every fuzzy metric space in the sense of Kramosil and Michalek has a completion which is unique up to an isometry.

From probabilistic metric spaces to fuzzy metric spaces

Remark

In a modern terminology [6, 3] a fuzzy metric (in the sense of Kramosil and Michalek) M on X is a fuzzy set on $X^2 \times [0, \infty[$ satisfying axioms (KM2)-(KM5), being $*$ a continuous t-norm and where (KM1) is replaced by

(KM1)' $M(x, y, 0) = 0$

Now, essentially because (KM6) has been removed, in this case a fuzzy metric cannot be regarded as a Menger space.
Nevertheless, in the same way than in the Menger spaces theory, a topology τ_M deduced from M is defined on X, and the concepts relative to completeness in *PM* spaces can be translated to the fuzzy theory.

Fuzzy metric spaces. Preliminaries

The concept of fuzzy metric space we deal with is due to George and Veeramani [3] and the axiomatic of this theory is exposed as follows.

Definition

([3]). A fuzzy metric space is an ordered triple $(X, M, *)$ *such that* X *is a (nonempty) set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X \times X \times]0, +\infty[$ *satisfying the following conditions, for all* $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, +\infty[\to]0, 1]$ *is continuous.*

$M(x, y, t)$ is considered as the degree of nearness of x and y with respect to t.
The axiom (GV1) is justified by the authors because in the same way that a classical metric cannot take the value ∞ then M cannot take the value 0.
The axiom (GV2) is equivalent to the following:

$$M(x, x, t) = 1 \text{ for all } x \in X \text{ and } t > 0, \text{ and } M(x, y, t) < 1 \text{ for all } x \neq y \text{ and } t > 0.$$

The axiom (GV2) gives the idea that when $x = y$ the degree of nearness of x and y is *perfect*, or simply 1, and then $M(x, x, t) = 1$ for each $x \in X$ and for each $t > 0$.

Fuzzy metric spaces. Preliminaries

Axioms (GV3)-(GV4) coincide with (KM3)-(KM4), respectively.
Finally, in (GV5) the authors assume that the variable t behave nicely, that is assume that for fixed x and y, $t \longrightarrow M(x, y, t)$ is a continuous function. Accordingly to the terminology of PM spaces we will denote this function by M_{xy}.
From now on by a fuzzy metric space we mean a fuzzy metric space in the sense of George and Veeramani. Contrary to that what happens with the (original) spaces of Kramosil and Michalek, these spaces cannot be regarded as PM spaces.

Lemma

Let X be a non-empty set. If $(M, *)$ is a fuzzy metric on X and $\diamond$ is a continuous t-norm such that $\diamond \leq *$, then $(M, \diamond)$ is a fuzzy metric on X. (The converse is false).

In consequence if $(M, \wedge)$ is a fuzzy metric on X, then $(M, *)$ is a fuzzy metric on X for all continuous t-norm $*$.
The following is a well-known result.

Lemma

The real function M_{xy} of Axiom (GV5) is increasing for all $x, y \in X$.

Fuzzy metric spaces. Preliminaries

Remark

If $(X, M, *)$ is a fuzzy metric space then it can be considered as a fuzzy metric space in the modern version of Kramosil and Michalek mentioned in Remark 1.6 defining $M(x, y, 0) = 0$, since (GV2) and (GV5) are stronger than (KM2) and (KM5), respectively.

Fuzzy metric spaces. Preliminaries

Let (X, d) be a metric space. Denote by $a \cdot b$ the usual product for all $a, b \in [0, 1]$, and let M_d be the fuzzy set defined on $X \times X \times]0, +\infty[$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(M_d, \cdot)$ is a fuzzy metric on X called *standard fuzzy metric* (see [3]).
The reader can notice the analogy between the next concepts and results stated for fuzzy metric spaces, with the corresponding ones in *PM* spaces. Their new quotes correspond to their appearance in the fuzzy setting.
George and Veeramani proved in [3] that every fuzzy metric M on X *generates* a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in]0, 1[$ and $t > 0$. It is also said thet τ_M is the topology *induced by M* or *deduced from M*.
A topological space (X, τ) is said to be *fuzzy metrizable* if there exists a fuzzy metric M on X compatible with τ, i.e. $\tau_M = \tau$.
If (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the fuzzy metric M_d ([3]). Consequently, every metrizable topological space is fuzzy metrizable.

Definition

A fuzzy metric M on X is said to be stationary, [10], if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$ and $B_M(x, \varepsilon)$ instead of $B_M(x, \varepsilon, t)$.

Examples of fuzzy metric spaces (Sapena, 2001)

Example

***(Fuzzy metrics deduced from metrics** [26]). Let (X, d) be a metric space, and denote $B(x, r)$ the open ball centered in $x \in X$ with radius $r > 0$.*
(i) For each $n \in \mathbb{N}$, $(X, M, \wedge)$ is a fuzzy metric space where M is given by

$$M(x, y, t) = \frac{1}{e^{\frac{d(x,y)}{t^n}}}, \quad \text{for all } x, y \in X, t > 0,$$

and $\tau_M = \tau(d)$.

(ii) For each $k, m \in \mathbb{R}^+, n \geq 1$, $(X, M, \wedge)$ is a fuzzy metric space where M is given by

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x,y)}, \quad \text{for all } x, y \in X, t > 0,$$

and $\tau_M = \tau(d)$.

Remark

The above expression of M cannot be generalized to $n \in \mathbb{R}^+$ (take the usual metric d on $\mathbb{R}$, $k = m = 1$, $n = 1/2$). Nevertheless is easy to verify that $(M, \cdot)$ is a fuzzy metric on X, for $n \geq 0$.

Examples of fuzzy metric spaces (Sapena, 2001)

Next, we will give fuzzy metrics which are not deduced explicitly from a metric.

Example

([32], Example 2.5.) Let $X = \mathbb{R}^+$. Define for $x, y \in X,\ t > 0$

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

Then $(M, \cdot)$ is a fuzzy metric on X.

Example

Let X be the real interval $]0, +\infty[,\ K \geq 0$ and $\alpha > 0$. It is easy to verify that $(X, M, \cdot)$ is a stationary fuzzy metric space,where M is defined by

$$M(x, y) = \left(\frac{\min\{x, y\} + K}{\max\{x, y\} + K}\right)^{\alpha} \qquad \textit{for all } x, y \in X.$$

$(X, M, \wedge)$ is not a fuzzy metric space. Indeed, for $\alpha = 1,\ K = 0$ if we take $x = 1, y = 2$ and $z = 3$, then
$M(x, z, t + s) = \frac{1}{3} < \min\{\frac{1}{2}, \frac{2}{3}\} = \min\{M(x, y, t), M(y, z, s)\}$.

Examples of fuzzy metric spaces (Sapena, 2001)

Next, we will give examples of fuzzy metric spaces for the t-norm $\mathfrak{L}$ which are not for the usual product.

Example

Let X be the real interval $]1, +\infty[$ and consider the mapping M on $X^2 \times]0, +\infty[$ given by

$$M(a, b) = 1 - \left(\frac{1}{a \wedge b} - \frac{1}{a \vee b}\right) \text{ for all } \ a, b \in X.$$

$(X, M, \mathfrak{L})$ is a stationary fuzzy metric space and $(X, M, \cdot)$ is not a fuzzy metric space. Further, the topology τ_M on X is the usual topology of $\mathbb{R}$ relative to X.

Example

Let X be the real interval $]2, +\infty[$ and consider the mapping M on $X^2 \times]0, +\infty[$ defined as follows

$$M(a, b) = \begin{cases} 1 & \text{if } a = b \\ \frac{1}{a} + \frac{1}{b} & \text{if } a \neq b \end{cases}$$

It is easy to verify that $(X, M, \mathfrak{L})$ is a stationary fuzzy metric space and $(X, M, \cdot)$ is not a fuzzy metric space.

Examples of fuzzy metric spaces (Sapena, 2001)

Example

Let $\{A, B\}$ be a nontrivial partition on the real interval $X =]2, +\infty[$. Define the mapping M on $X^2 \times]0, +\infty[$ as follows

$$M(x,y) = \begin{cases} 1 - \left(\frac{1}{x \wedge y} - \frac{1}{x \vee y}\right) & \text{if} \quad x, y \in A \quad \text{or} \quad x, y \in B \\ \\ \frac{1}{x} + \frac{1}{y} & \text{elsewhere} \end{cases}$$

Then $(X, M, \mathfrak{L})$ is a stationary fuzzy metric space and $(X, M, \cdot)$ is not a fuzzy metric space.

Definition

A subset A of X is said to be F-bounded if there exist $t > 0$ and $r \in]0, 1[$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Examples of fuzzy metric spaces (Sapena, 2001)

Proposition

If (X, d) is a metric space, then: $A \subset X$ is bounded in (X, d) if and only if it is F-bounded in $(X, M_d, *)$.

Proposition

Let $(X, M, *)$ be a fuzzy metric space and $k \in]0, 1[$. Define

$$N(x, y, t) = \max\{M(x, y, t), k\},$$

for each $x, y \in X$, $t > 0$. Then $(N, *)$ is an F-bounded fuzzy metric on X, which generates the same topology that M.

Proposition

Let $k > 0$. Suppose that $(X, M, *)$ is a fuzzy metric space where $*$ is one of the t-norms given above, and define:

$$N(x, y, t) = \frac{k + M(x, y, t)}{1 + k}$$

for all $x, y \in X, t > 0$.
Then, $(N, *)$ is an F-bounded fuzzy metric on X, which generates the same topology that M.

On non-Archimedean fuzzy metrics

Recall that a metric d on X is called non-Archimedean if
$d(x,z) \leq \max\{d(x,y), d(y,z)\}$, for all $x, y, z \in X$.

Definition

A fuzzy metric $(M,*)$ on X is called non-Archimedean if
$M(x,z,t) \geq \min\{M(x,y,t), M(y,z,t)\}$, *for all $x, y, z \in X, t > 0$.*

Proposition

Let d be a metric on X and M_d the corresponding standard fuzzy metric. Then, d is non-Archimedean if and only if M_d is non-Archimedean.

Recall that a completely regular space is called strongly zero-dimensional if each zero-set is the countable intersection of sets that are open and closed, and that a T_0 topological space (X,τ) is strongly zero-dimensional and metrizable if and only if there is a uniformity $\mathcal{U}$ compatible with τ that has a countable transitive base.

Theorem

A topological space (X,τ) is strongly zero-dimensional and metrizable if and only if (X,τ) is non-Archimedeanly fuzzy metrizable.

Metrizability of fuzzy metric spaces (Gregori-Romaguera, 2002)

Uniform structure in the fuzzy metric space $(X, M, *)$

Let $(X, M, *)$ be a fuzzy metric space. For each $n \in \mathbb{N}$ define:

$$U_n = \{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}.$$

The (countable) family $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}_M$ on X such that the topology induced by $\mathcal{U}_M$ agrees with the topology induced by the fuzzy metric M. The uniformity $\mathcal{U}_M$ will be called the uniformity deduced from M or generated by M. Applying the Kelley's metrization lemma the following results hold.

Lemma

Let $(X, M, *)$ be a fuzzy metric space. Then, (X, τ_M) is a metrizable topological space.

Corollary

A topological space is metrizable if and only if it admits a compatible fuzzy metric.

Corollary

Every separable fuzzy metric space is second countable.

Metrizability of fuzzy metric spaces (Gregori-Romaguera, 2002)

Definition (George and Veeramani, 1995)

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence (or M-Cauchy), if for each $\varepsilon \in]0,1[$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $m, n \geq n_0$.

Proposition

$\{x_n\}$ is a $d-$Cauchy sequence (i.e., a Cauchy sequence in (X, d)) if and only if it is a Cauchy sequence in $(X, M_d, *)$.

Let us recall that a metrizable topological space (X, τ) is said to be completely metrizable if it admits a complete metric. On the other hand, a fuzzy metric space $(X, M, *)$ is called complete if every Cauchy sequence is convergent. If $(X, M, *)$ is a complete fuzzy metric space, we say that M is a complete fuzzy metric on X.

Theorem

Let $(X, M, *)$ be a complete fuzzy metric space. Then, (X, τ_M) is completely metrizable.

Corollary

A topological space is completely metrizable if and only it admits a compatible complete fuzzy metric.

Compactness of fuzzy metric spaces (Gregori and Romaguera, 2002)

Definition

A fuzzy metric space $(X, M, *)$ is called precompact if for each r, with $0 < r < 1$, and each $t > 0$, there is a finite subset A of X, such that $X = \bigcup_{a \in A} B(a, r, t)$. In this case, we say that M is a precompact fuzzy metric on X.

A fuzzy metric space $(X, M, *)$ is called *compact* if (X, τ_M) is a compact topological space.

Lemma

A fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence.

Theorem

A fuzzy metric space $(X, M, *)$ is separable if and only if (X, τ_M) admits a compatible precompact fuzzy metric.

Compactness of fuzzy metric spaces (Gregori and Romaguera, 2002)

Lemma

Let $(X, M, *)$ be a fuzzy metric space. If a Cauchy sequence clusters to a point $x \in X$, then the sequence converges to x.

Theorem

A fuzzy metric space is compact if and only it is precompact and complete.

Theorem

A metrizable topological space is compact if and only every compatible fuzzy metric is precompact.

Theorem

A metrizable topological space is compact if and only every compatible fuzzy metric is complete.

Continuity and uniform continuity (Gregori, Romaguera, Sapena, 2001)

Let us recall that a uniformity $\mathcal{U}$ on a set X has the Lebesgue property provided that for each open cover $\mathcal{G}$ of X there is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{G}$, and $\mathcal{U}$ is said to be equinormal if for each pair of disjoint nonempty closed subsets A and B of X there is $U \in \mathcal{U}$ such that $U(A) \cap B = \emptyset$. A metric d on X has the Lebesgue property provided that the uniformity $\mathcal{U}_d$, induced by d, has the Lebesgue property and d is equinormal provided that $\mathcal{U}_d$ so is.
Let (X, M) and (Y, N) be fuzzy metric spaces. In [4] is given the following definition.

Definition

A mapping from X to Y is said to be uniformly continuous if for each $\varepsilon \in]0, 1[$ and each $t > 0$, there exist $r \in]0, 1[$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - r$.

It is easy to verify that this definition is equivalent to consider $f : (X, \mathcal{U}_M) \to (Y, \mathcal{U}_N)$ as uniform continuous with respect to the uniformities $\mathcal{U}_M$ and $\mathcal{U}_N$ deduced from M and N respectively, and then it is continuous from (X, τ_M) to (Y, τ_N).
Similarly to the classical metric case, if $f : (X, M) \to (Y, N)$ is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ is a Cauchy sequence in Y.

Definition

We say that a real valued function f on the fuzzy metric space $(X, M, *)$ is $\mathbb{R}$-uniformly continuous provided that for each $\varepsilon > 0$ there exist $r \in]0, 1[$ and $s > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $M(x, y, s) > 1 - r$.

Continuity and uniform continuity (Gregori, Romaguera, Sapena, 2001)

Definition

A fuzzy metric $(M, *)$ on a set X is called equinormal if for each pair of disjoint nonempty closed subsets A and B of (X, τ_M) there is $s > 0$ such that $\sup\{M(a, b, s) : a \in A,\ b \in B\} < 1$

Definition

We say that a fuzzy metric $(M, *)$ on a set X has the Lebesgue property if for each open cover $\mathcal{G}$ of (X, τ_M) there exist $r \in]0, 1[$ and $s > 0$ such that $\{B_M(x, r, s) : x \in X\}$ refines $\mathcal{G}$.

Remark

Notice that if (X, d) is a metric space, then the fuzzy metric $(M_d, *)$ has the Lebesgue property (resp. is equinormal) if and only if d has the Lebesgue property (resp. is equinormal).

Continuity and uniform continuity (Gregori, Romaguera, Sapena 2001)

In [13] fuzzy metric spaces for which real valued continuous functions are uniformly continuous, were characterized as follows.

Theorem

For a fuzzy metric space $(X, M, *)$ the following are equivalent.

(1) *For each fuzzy metric space $(Y, N, \star)$ any continuous mapping from (X, τ_M) to (Y, τ_N) is uniformly continuous as a mapping from $(X, M, *)$ to $(Y, N, \star)$.*

(2) *Every real valued continuous function on (X, τ_M) is $\mathbb{R}$-uniformly continuous on $(X, M, *)$.*

(3) *Every real valued continuous function on (X, τ_M) is uniformly continuous on $(X, \mathcal{U}_M)$.*

(4) *$(M, *)$ is an equinormal fuzzy metric on X.*

(5) *$\mathcal{U}_M$ is an equinormal uniformity on X.*

(6) *The uniformity $\mathcal{U}_M$ has the Lebesgue property.*

(7) *The fuzzy metric $(M, *)$ has the Lebesgue property.*

Continuity and uniform continuity (Gregori, Romaguera, Sapena, 2004)

In order to state new versions of the classical Banach Contraction Principle for fuzzy metric spaces, in [7] the authors gave a concept of t-uniformly continuous function, closer to a concept of contractive mapping, as follows.

Definition

A mapping f from a fuzzy metric space (X, M) to a fuzzy metric space (Y, N) is called t-uniformly continuous if for each $\varepsilon \in]0, 1[$ and each $t > 0$, there exists $r \in]0, 1[$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, t) > 1 - r$.

It was proved in [4] that every continuous mapping form a compact fuzzy metric space to a fuzzy metric space is uniformly continuous. This result was improved in [14] as follows.

Proposition

Every continuous mapping from a compact fuzzy metric space $(X, M, *)$ to a fuzzy metric space $(Y, N, *)$ is t-uniformly continuous.

In [14] Example 1 an example of a uniformly continuous mapping which is not t-uniformly continuous was given.

Continuity and uniform continuity (Gregori, López-Crevillén, Morillas, 2009)

As in the classical case the next theorem is satisfied [19].

Theorem

Let $(X, M, *)$ and $(Y, N, \star)$ be two fuzzy metric spaces, D a dense subspace of X and $f : D \longrightarrow Y$ a uniformly continuous mapping. Suppose Y complete. Then, it exists a unique mapping $g : X \longrightarrow Y$ uniformly continuous that extends f. Further, if f is t-uniformly continuous, then g is t-uniformly continuous.

Continuity and uniform continuity (Gregori, Romaguera, Sapena, 2004)

In [14] those fuzzy metric spaces for which real-valued continuous functions are t-uniformly continuous where characterized as follows.

Definition

A fuzzy metric $(M, *)$ on a set X is called t-equinormal if for each pair of disjoin nonempty closed subsets A and B of (X, τ_d) and each $t > 0$,
$\sup\{M(a, b, t) : a \in A,\ b \in B\} < 1$.

Theorem

For a fuzzy metric space $(X, M, *)$ the following are equivalent.

(1) *For each fuzzy metric space $(Y, N, \star)$ any continuous mapping from (X, τ_M) to (Y, τ_N) is t-uniformly continuous as a mapping form $(X, M, *)$ to $(Y, N, \star)$.*

(2) *Any real-valued continuous function on (X, τ_M) is t-uniformly continuous from $(X, M, *)$ to the Euclidean fuzzy metric space $(\mathbb{R}, M_{|\cdot|}, \cdot)$.*

(3) *The fuzzy metric $(M, *)$ is t-equinormal.*

On completion of fuzzy metric spaces (Gregori and Romaguera, 2002)

Given a metric space (X, d) we shall denote by $(\widetilde{X}, \widetilde{d})$ the (metric) completion of (X, d).
In a first attempt to obtain a satisfactory notion of fuzzy metric completion we start by analyzing the relationship between the standard fuzzy metrics of d and $\widetilde{d}$, respectively.

Example

Let (X, d) be a metric space and let f be an isometry from (X, d) onto a dense subspace of $(\widetilde{X}, \widetilde{d})$. The standard fuzzy metric $(M_{\widetilde{d}}, \cdot)$ of $\widetilde{d}$ is given by

$$M_{\widetilde{d}}(\widetilde{x}, \widetilde{y}, t) = \frac{t}{t + \widetilde{d}(\widetilde{x}, \widetilde{y})}$$

for all $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ and $t > 0$. Hence, we have $M_d(x, y, t) = M_{\widetilde{d}}(f(x), f(y), t)$ for all $x, y \in X$ and $t > 0$.

On completion of fuzzy metric spaces (Gregori and Romaguera, 2002)

The preceding example agrees with the following natural notions.

Definition

Let $(X, M, *)$ and $(Y, N, \star)$ be two fuzzy metric spaces. A mapping f from X to Y is called an isometry if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.

As in the classical metric case, it is clear that every isometry is one-to-one.

Definition

Two fuzzy metric spaces $(X, M, *)$ and $(Y, N, \star)$ are called isometric if there is an isometry from X onto Y.

Definition

Let $(X, M, *)$ be a fuzzy metric space. A fuzzy metric completion of $(X, M, *)$ is a complete fuzzy metric space $(Y, N, \star)$ such that $(X, M, *)$ is isometric to a dense subspace of Y.

On completion of fuzzy metric spaces (Gregori and Romaguera, 2002)

Next we show that unfortunately there exists a fuzzy metric space that does not admit any fuzzy metric completion in the sense of Definition 8.4.

Example

Let $(x_n)_{n=3}^{\infty}$ and $(y_n)_{n=3}^{\infty}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$.

Put $X = A \cup B$. Define a real valued function M on $X \times X \times (0, \infty)$ as follows:

$$\begin{aligned} M(x_n, x_m) &= M(y_n, y_m) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right], \\ M(x_n, y_m) &= M(y_m, x_n) = \frac{1}{n} + \frac{1}{m}, \end{aligned}$$

for all $n, m \geq 3$. Then $(X, M, \mathfrak{L})$ a stationary fuzzy metric space.

In [9] it it proved that $(M, *)$ is a not complete fuzzy metric on X, which has not a fuzzy metric completion.

Characterizing completable fuzzy metric spaces (Gregori and Romaguera, 2003)

Definition

Let $(X, M, *)$ be a fuzzy metric space. Then a pair $(a_n)_n, (b_n)_n$, of Cauchy sequences in X, is called:
a) point-equivalent if there is $s > 0$ such that $\lim_n M(a_n, b_n, s) = 1$.
b) equivalent, denoted by $(a_n)_n \sim (b_n)_n$, if $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Theorem

A fuzzy metric space $(X, M, *)$ is completable if and only if it satisfies the two following conditions:

(C1) *Given two Cauchy sequences $(a_n)_n, (b_n)_n$, in X, then*

$$t \mapsto \lim_n M(a_n, b_n, t)$$

is a continuous function on $(0, +\infty)$ with values in $(0, 1]$.

(C2) *Each pair of point-equivalent Cauchy sequences is equivalent.*

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

In [22] the author gave the following definition.

Definition

([22]) Let (X, M) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be point convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$.

In such a case we say that $\{x_n\}$ is p-convergent to x_0 for $t_0 > 0$, or, simply, $\{x_n\}$ is p-convergent.
Equivalently, $\{x_n\}$ is p-convergent if there exist $x_0 \in X$ and $t_0 > 0$ such that $\{x_n\}$ is eventually in $B(x_0, r, t_0)$ for each $r \in]0, 1[$ (or, without lost of generality, in $B(x_0, \frac{1}{n}, t_0)$ for each $n \in \mathbb{N}$).
Clearly $\{x_n\}$ is convergent to x_0 if and only if $\{x_n\}$ is p-convergent to x_0 for all $t > 0$.
The following properties hold [22]:

(1) If $\lim_n M(x_n, x, t_1) = 1$ and $\lim_n M(x_n, y, t_2) = 1$ then $x = y$.

(2) If $\lim_n M(x_n, x_0, t_0) = 1$ then $\lim_k M(x_{n_k}, x_0, t_0) = 1$ for each subsequence (x_{n_k}) of $\{x_n\}$.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

By property (1) the next Corollary is obtained:

Corollary

If $\{x_n\}$ is p-convergent to x_0 and it is convergent, then $\{x_n\}$ converges to x_0.

Corollary

Let (X, M) be a completable fuzzy metric space. If $\{x_n\}$ is a Cauchy sequence in X, and it is p-convergent to $x_0 \in X$, then $\{x_n\}$ converges to x_0.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

An example of a p-convergent sequence which is not convergent is given in the next example.

Example

([22]) Let $\{x_n\} \subset]0,1[$ *be a strictly increasing sequence convergent to 1 respect to the usual topology of* $\mathbb{R}$ *and* $X = \{x_n\} \cup \{1\}$. *Define on* $X^2 \times \mathbb{R}^+$ *the function M given by* $M(x,x,t) = 1$ *for each* $x \in X,\ t > 0$, $M(x_n, x_m, t) = \min\{x_n, x_m\}$, *for all* $m, n \in \mathbb{N},\ t > 0$, *and* $M(x_n, 1, t) = M(1, x_n, t) = \min\{x_n, t\}$ *for all* $n \in \mathbb{N},\ t > 0$. *Then* $(M, *)$ *is a fuzzy metric on X, where* $a * b = \min\{a, b\}$. *The sequence* $\{x_n\}$ *is not convergent since* $\lim_n M(x_n, 1, \frac{1}{2}) = \frac{1}{2}$. *Nevertheless it is p-convergent to 1, since* $\lim_n M(x_n, 1, 1) = 1$.

Notice that in the above example $\{1\}$ is open in τ_M since $B(1, \frac{1}{2}, \frac{1}{2}) = \{1\}$. On the other hand for $r \in]0,1[$ we have that $B(1, r, 1) = \{x_m, x_{m+1}, \ldots\} \cup \{1\}$ where x_m is the first element of $\{x_n\}$ such that $0 < 1 - r < x_m$. Hence, the family of open balls $\{B(1, r, 1) : r \in]0,1[\}$ is not a local base at 1. This fact motivates our next definition.

Definition

We say that the fuzzy metric space $(X, M, *)$ *is principal (or simply, M is principal) if* $\{B(x, r, t) : r \in]0,1[\}$ *is a local base at* $x \in X$, *for each* $x \in X$ *and each* $t > 0$.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

As we have just seen, the fuzzy metric of Example 10.4 is not principal. Next we see some examples of principal fuzzy metrics.

Example

(a) Stationary fuzzy metrics are, obviously, principal.

(b) The well-known standard fuzzy metric is principal.

(c) $M(x,y,t) = e^{-\frac{d(x,y)}{t}}$*, where* d *is a metric on* X*, [3], is principal.*

(d) $M(x,y,t) = \dfrac{\min\{x,y\}+t}{\max\{x,y\}+t}$ *is a fuzzy metric on* $\mathbb{R}^+$*, [32], which is principal.*

Theorem

The fuzzy metric space (X,M) *is principal if and only if all p-convergent sequences are convergent.*

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

Next we give an example of a complete fuzzy metric space which is not principal.

Example

Let $X = \mathbb{R}^+$ and let $\varphi : \mathbb{R}^+ \to]0,1]$ be a function given by $\varphi(t) = t$ if $t \leq 1$ and $\varphi(t) = 1$ elsewhere. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} 1 & x = y \\ \dfrac{\min\{x,y\}}{\max\{x,y\}} \cdot \varphi(t) & x \neq y \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X and, since $M(x,y,t) < t$, whenever $t \in]0,1[$ and $x \neq y$, it is obvious that the only Cauchy sequences in X are the constant sequences and so, X is complete.

This fuzzy metric is not principal. In fact, notice that $B(x, \frac{1}{2}, \frac{1}{2}) = \{x\}$ for each $x \in X$ and so τ_M is the discrete topology. Now, if we set $x = 1$ and $t = 1$ we have $B(1,r,1) =]1-r, \frac{1}{1-r}[$ for all $r \in]0,1[$ and so $\{B(1,r,1) : r \in]0,1[\}$ is not a local base at $x = 1$, since $\{1\}$ is open.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

In the next example, $(X, M, \cdot)$ is a fuzzy metric space which is not principal and not completable [17].

Example

Let $X =]0,1]$, $A = X \cap \mathbb{Q}$, $B = X \setminus A$. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} \dfrac{\min\{x,y\}}{\max\{x,y\}} & x, y \in A \text{ or } x, y \in B,\ t > 0 \\[2ex] \dfrac{\min\{x,y\}}{\max\{x,y\}} & x \in A,\ y \in B \text{ or } x \in B,\ y \in A,\ t \geq 1 \\[2ex] \dfrac{\min\{x,y\}}{\max\{x,y\}} \cdot t & \text{elsewhere} \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X. It is proved in [17] that M is not principal.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

Continuing the above study we give the next definition.

Definition

Let (X, M) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be p-Cauchy if for each $\varepsilon \in]0, 1[$ there are $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that $M(x_n, x_m, t_0) > 1 - \varepsilon$ for all $n, m \geq n_0$, i.e. $\lim_{m,n} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

In such a case we say that $\{x_n\}$ is p-Cauchy for $t_0 > 0$, or, simply, $\{x_n\}$ is p-Cauchy. Clearly $\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ is p-Cauchy for all $t > 0$ and, obviously, p-convergent sequences are p-Cauchy.

Definition

The fuzzy metric space (X, M) is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X. In such a case M is called p-complete.

Obviously, p-completeness and completeness are equivalent concepts in stationary fuzzy metrics, and it is easy to verify that the standard fuzzy metric M_d is p-complete if and only if M_d is complete.

Principal fuzzy metrics (Gregori, López-Crevillén, Morillas, Sapena, 2009)

Proposition

Let (X, M) be a principal fuzzy metric space. If X is p-complete then X is complete.

The assumption that X is principal cannot be removed in the last proposition as shows the next example.

Example

Consider the fuzzy metric space $(X, M, *)$ of Example 10.4. The sequence $\{x_n\}$ satisfies $\lim_{m,n} M(x_n, x_m, t) = 1$ for all $t > 0$, so $\{x_n\}$ is a Cauchy sequence and in consequence X is not complete, since $\{x_n\}$ is not convergent.
Next we show that X is p-complete.
Let $\{x_n\}$ be a p-Cauchy sequence in X. Then, with an easy argument one can verify that $\{x_n\}$ must be a convergent sequence to 1 with respect to the usual topology of $\mathbb{R}$ relative to X. Now, $\lim_n M(x_n, 1, 1) = \lim_n x_n = 1$ and hence $\{x_n\}$ is p-convergent to 1.

One could expect p-Cauchy sequences to be Cauchy sequences in principal fuzzy metric spaces. In fact, this property is satisfied by all examples of Example 10.6. Nevertheless, as shown the authors in [17] it is not true, in general, for any principal fuzzy metric and, in consequence, the converse of the above proposition is not true.

On continuity and t-continuity (Gregori, López-Crevillén and Morillas (2009)

The definition of continuity of a mapping f from a fuzzy metric space (X, M) to a fuzzy metric space (Y, N) can be given using four parameters as follows.
f is continuous at $x_0 \in X$ if given $\varepsilon \in]0, 1[$ and $t > 0$ there exist $\delta \in]0, 1[$ and $s > 0$ such that $M(x_0, x, s) > 1 - \delta$ implies $N(f(x_0, f(x), t) > 1 - \varepsilon$.
Obviously the condition of continuity of a mapping f between stationary fuzzy metric spaces only needs two parameters. Then, thinking in stationary fuzzy metric spaces and according to the concept of t-uniformly continuous function we give the next definition, by mean of three parameters.

Definition

We will say that a mapping f from the fuzzy metric space (X, M) to a fuzzy metric space (Y, N) is t-continuous at $x_0 \in X$ if given $\varepsilon \in]0, 1[$ and $t > 0$ there exists $\delta \in]0, 1[$ such that $M(x_0, x, t) > 1 - \delta$ implies $N(f(x_0), f(x), t) > 1 - \varepsilon$.

We will say that f is t-continuous on X if it is t-continuous at each point of X.
If M is a stationary fuzzy metric then each continuous mapping is t-continuous.
Obviously if f is t-continuous at x_0 then f is continuous at x_0. The converse is false, as shown the authors in [19].
It is obvious that each t-uniformly continuous mapping is t-continuous. The converse is false. We will see in the next example a t-continuous mapping (and uniformly continuous) which is not t-uniformly continuous.

On continuity and t-continuity (Gregori, López-Crevillén and Morillas (2009)

Example

Let $X = \{1, 2, 3, \cdots\}$. Consider on X the fuzzy metric M, for the usual product, given by

$$M(m,n,t) = \begin{cases} \dfrac{\min\{m,n\}}{\max\{m,n\}} \cdot t & m \neq n,\ t < 1 \\ \dfrac{\min\{m,n\}}{\max\{m,n\}} & \text{elsewhere} \end{cases}$$

τ_M is the discrete topology on X.
Every function $f : X \longrightarrow \mathbb{R}$ is uniformly continuous for any fuzzy metric on $\mathbb{R}$.
Every function $f : X \longrightarrow \mathbb{R}$ is t-continuous for any fuzzy metric N on $\mathbb{R}$.
Now, consider the mapping $f : X \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$$

Consider the fuzzy metric M on X and the standard fuzzy metric $M_{|\cdot|}$ on $\mathbb{R}$. We will see that f is not t-uniformly continuous for these fuzzy metrics.
Let $t = 1$ and $\varepsilon = 0.5$. For every $\delta \in]0,1[$ there exist $n \in X$ such that $\dfrac{n}{n+1} > 1 - \delta$ and so $M(n, n+1, t) = \dfrac{n}{n+1} > 1 - \delta$. Therefore $M_{|\cdot|}(f(n), f(n+1), t) = \dfrac{1}{1+1} = \dfrac{1}{2}$ and so f is not t-uniformly continuous.

On continuity and t-continuity (Gregori, López-Crevillén and Morillas (2009)

The next example shows a larger class than stationary fuzzy metrics in which continuous functions are t-continuous.

Proposition

Let f be a mapping from the fuzzy metric space (X, M) to the fuzzy metric space (Y, N), continuous at x_0. If M is principal then f is t-continuous at x_0.

If (X, d) is a metric space then the standard fuzzy metric M_d is principal [17] so we have the next corollary.

Corollary

A mapping f from the fuzzy metric space (X, M_d) to a fuzzy metric space (Y, N) is continuous at x_0 if and only if f is t-continuous at x_0.

Strong fuzzy metrics (Morillas and Sapena, 2009)

If $(M, \wedge)$ is a fuzzy metric on X then the triangular inequality (GV4) becomes for all $x, y, z \in X$ and $t, s > 0$

$$M(x, z, t+s) \geq M(x, y, t) \wedge M(y, z, s)$$

Now if we demand also that

$$M(x, z, t) \geq M(x, y, t) \wedge M(y, z, t)$$

we obtain the notion of fuzzy ultrametric (non-Archimedean fuzzy metric [26]).
Then it suggests the following definition.

Definition

[18] Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is said to be strong if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \tag{GV4'}$$

Notice that this axiom (GV4') cannot replace axiom (GV4) in the definition of fuzzy metric since in that case M could not be a fuzzy metric on X.
It is possible to define a strong fuzzy metric by replacing (GV4) by (GV4') and demanding in (GV5) that the function $M_{x,y}$ be an increasing continuous function on t, for each $x, y \in X$. (Indeed, in such a case we have that $M(x, z, t+s) \geq M(x, y, t+s) * M(y, z, t+s) \geq M(x, y, t) * M(y, z, s)$).

Strong fuzzy metrics (Morillas and Sapena, 2009)

Example

(a) *Stationary fuzzy metrics are strong.*

(b) *Fuzzy ultrametrics are strong.*

(c) *The fuzzy metric* $(M, \cdot)$ *on* $\mathbb{R}^+$ *defined by* $M(x, y, t) = \dfrac{\min\{x, y\} + t}{\max\{x, y\} + t}$ *is strong.*

In the next examples d *is a metric on* X*.*

(d) *Let* $\varphi : \mathbb{R}^+ \rightarrow]0, 1]$ *be an increasing continuous function. Define the function* M *on* $X^2 \times \mathbb{R}^+$ *by*

$$M(x, y, t) = \frac{\varphi(t)}{\varphi(t) + d(x, y)}$$

Then $(M, \cdot)$ *is strong. In particular, the well-known standard fuzzy metric* M_d *is strong for the usual product.*

(e) *The function* M *on* $X^2 \times \mathbb{R}^+$ *given by* $M(x, y, t) = e^{-\frac{d(x,y)}{t}}$ *is strong for the usual product.*

(f) *The standard fuzzy metric* M_d *is a fuzzy ultrametric (and so it is strong for the t-norm minimum) if and only if* d *is an ultrametric [26].*

(g) *If* d *is a metric which is not ultrametric then* $(M_d, \wedge)$ *is a non-strong fuzzy metric on* X*.*

Strong fuzzy metrics (Morillas and Sapena, 2009)

Next results can be seen in [18]. Let $(M,*)$ be a non-stationary strong fuzzy metric. We define the family of functions $\{M_t : t \in \mathbb{R}^+\}$ where $M_t : X^2 \longrightarrow]0,1]$ is given by $M_t(x,y) = M(x,y,t)$. With this notation we have the following proposition.

Proposition

Let $(M,*)$ be a non-stationary fuzzy metric on X. Then:

(i) *$(M,*)$ is strong if and only if $(M_t,*)$ is a stationary fuzzy metric on X for each $t \in \mathbb{R}^+$.*

(ii) *If $(M,*)$ is strong then $\tau_M = \bigvee\{\tau_{M_t} : t \in \mathbb{R}^+\}$.*

Strong fuzzy metrics (Morillas and Sapena, 2009)

If M is a strong fuzzy metric we will say that $\{M_t : t \in \mathbb{R}^+\}$ is the family of stationary fuzzy metrics deduced from M.

Example

(a) *Let d be a metric on X. Then $(M_{d_t}, \wedge)$ is a stationary fuzzy metric on X for each $t > 0$ if and only if $(M_d, \wedge)$ is strong if and only if d is an ultrametric on X.*

(b) *Consider the strong fuzzy metric M of Example 12.2 (c). Then,*

$$M_t(x,y) = \frac{\min\{x,y\} + t}{\max\{x,y\} + t}$$

is a stationary fuzzy metric for each $t > 0$ and it is easy to verify that $\tau_{M_t} = \tau_M$ for each $t > 0$.

Strong fuzzy metrics (Morillas and Sapena, 2009)

Example

(c) *Consider the function M on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ given by*

$$M(x,y,t) = \begin{cases} 1 & x = y \\ \dfrac{\min\{x,y\}}{\max\{x,y\}} \cdot \varphi(t) & x \neq y \end{cases}$$

where $\varphi(t) = \begin{cases} t & 0 < t \leq 1 \\ 1 & t > 1 \end{cases}$.

It is easy to verify that $(M, \cdot)$ is a strong fuzzy metric on $\mathbb{R}^+$ and τ_M is the discrete topology on $\mathbb{R}^+$.

For $t \geq 1$ we have that $M_t(x,y) = \dfrac{\min\{x,y\}}{\max\{x,y\}}$ *and τ_{M_t} is the usual topology of $\mathbb{R}$ relative to $\mathbb{R}^+$.*

For $t < 1$ we have that $M_t(x,y) = \dfrac{\min\{x,y\}}{\max\{x,y\}} \cdot t$ *and so τ_{M_t} is the discrete topology.*

Strong fuzzy metrics (Morillas and Sapena, 2009)

Now it arises the natural question of when a family $(M_t, *)$ of stationary fuzzy metrics on X for $t \in \mathbb{R}^+$, defines a fuzzy metric $(M, *)$ on X by means of the formula $M(x, y, t) = M_t(x, y)$ for each $x, y \in X,\ t \in \mathbb{R}^+$. The next proposition answers this question.

Proposition

Let $\{(M_t, *) : t \in \mathbb{R}^+\}$ be a family of stationary fuzzy metrics on X.

(i) *The function M on $X^2 \times \mathbb{R}^+$ defined by $M(x, y, t) = M_t(x, y)$ is a fuzzy metric on X when considering the t-norm $*$, if and only if $\{M_t : t \in \mathbb{R}^+\}$ is an increasing family (i.e. $M_t \leq M_{t'}$ if $t < t'$) and the function $M_{xy} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $M_{xy}(t) = M_t(x, y)$ is a continuous function, for each $x, y \in X$.*

(ii) *If conditions of (i) are fulfilled then $(M, *)$ is strong and $\{(M_t, *) : t \in \mathbb{R}^+\}$ is the family of stationary fuzzy metrics deduced from M.*

By (ii) we can notice that a strong fuzzy metric is characterized by its family $\{M_t : t \in \mathbb{R}^+\}$ of stationary fuzzy metrics.

Strong fuzzy metrics (Morillas and Sapena, 2009)

An easy consequence of the previous definitions is the next proposition.

Proposition

Let $\{M_t : t \in \mathbb{R}^+\}$ be the family of stationary fuzzy metrics deduced from the strong fuzzy metric M on X. Then the sequence $\{x_n\}$ in X is M-Cauchy if and only if $\{x_n\}$ is M_t-Cauchy for each $t > 0$.

Corollary

Let $(X, M, *)$ be a strong fuzzy metric space. $(X, M, *)$ is complete if and only if $(X, M_t, *)$ is complete for each $t \in \mathbb{R}^+$.

Other fuzzy metric topics

- **Ascoli-Arzela theorem for fuzzy metric spaces (George and Veeramni, 1995)**
- **Uniform continuity and contractivity (Gregori and Sapena, 2001)**
- **Fixed point theorems (Gregori, Sapena, 2001)**
- **Fixed point theorem in Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense (Gregori and Sapena, 2001)**
- **The construction of the Hausdorff metric on $\mathcal{K}_0(X)$ (J. Rodríguez-López, S. Romaguera, 2004)**
- **Fuzzy quasi-metric spaces (Gregori and Romaguera, 2004)**
- **On bicompletion of fuzzy quasi-metric spaces (Gregori, Romaguera and Sapena, 2004)**
- **The Doitchinov completion of fuzzy quasi-metric spaces (Gregori, Mascarell and Sapena, 2005)**
- **Intuitionistic fuzzy metric spaces (JH. Park, 2004) (Gregori, Romaguera, Veeramani, 2006)**
- **Application of fuzzy metrics to color image filtering (Morillas, Gregori, Peris-Fajarnés and Latorre, 2005**

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