(Preliminary) program for the Advanced Course on Topological Quantum Field Theories

University of Almería (Spain), 19-23 October 2009

October 8, 2009

1 Frobenius algebras and 2D TQFTs, by Joachim Kock

1. First two lectures: heuristics and formal definition of topological quantum field theory; the symmetric monoidal category nCob of cobordisms in dimension n; generators-and-relations presentation of 2Cob as a symmetric monoidal category.
2. Last two lectures: Introduction to Frobenius algebras, classical theory and graphical calculus; the 2D theorem characterising 2D TQFTs as commutative Frobenius algebras; some examples; some more categorical yoga, and the universal property of 2Cob.

2 TFT, RCFT and all that, by Christoph Schweigert

1. In the first lecture, we reformulate the axioms of a two-dimensional topological field theory and present lattice topological field theories as examples.
2. In the second lecture, we review the compactified free boson as a motivating example for a two-dimensional rational conformal field theory.
3. I the last two lectures we formalize these lectures as a categorification of the structure of a lattice topological field theory.

3 TQFTs, Operads and Moduli Spaces, by Andrey Lazarev

1. Formal structure of QFT and asymptotic expansion of finite-dimensional integrals.

In this lecture we give an abstract formulation of a quantum field theory and consider its finite-dimensional model corresponding to the discrete spacetime. We show how the path integral formulation of QFT naturally leads to sums over Feynman diagrams.

2. Operads and topological conformal field theories.

   We describe how a string propagating in spacetime naturally leads one to consider Frobenius algebras; this is a simplest example of a topological conformal field theory. More sophisticated examples take into account the topology of the moduli spaces of Riemann surfaces. The underlying algebraic structure is that of an algebra over an operad and we discuss this notion in some detail.

3. Graph complexes.

   This lecture is devoted to graph complexes introduced by Kontsevich in the beginning of the 90's. These complexes are beautiful combinatorial structures; they are easy to define but very hard to compute with. We explain their relation to the homology of some infinite-dimensional Lie algebras and related geometric objects.

4. Modular operads.

   A modular operad is a notion introduced by Getzler and Kapranov some 10 years ago; it gives a proper homework encompassing graph complexes on the one hand and operads on the other. The most modular operads come from moduli spaces of various sort, hence the name. We consider examples and, if time permits, describe Feynman transforms of modular operads and algebras over them.

References for Lazarev's lecture series:

[CL07] J. Chuang, A. Lazarev. Dual Feynman transform for modular operads. Communications in Number Theory and Physics, 1 (2007), 605-649.

[E98] P. Etingof. Mathematical ideas and notions of quantum field theory. Web notes.

[GK98] E. Getzler, M. Kapranov. Modular Operads. Compositio Math. 110 (1998), no. 1, 65–126.

[GK94] V. Ginzburg, M. Kapranov. Koszul duality for operads. Duke Math. J. 76 (1994), no. 1, 203–272.

[K02] J. Kock. Frobenius Algebras and 2D Topological Quantum field Theories. Cambridge University Press, Cambridge 2003.

[Kon93] M. Kontsevich. Formal noncommutative symplectic geometry. The Gelfand Mathematical Seminars, 1990-1992, pp. 173–187, Birkhauser Boston, Boston, MA, 1993.

[Kon94] M. Kontsevich. Feynman diagrams and low-dimensional topology. First European Congress of Mathematics, Vol 2 Paris, 1992, 97-121, Progr. Math., 120, Birkhauser, Basel, 1994.

[HL09] A. Hamilton, A. Lazarev. Noncommutative geometry and cohomology theories for homotopy algebras, Alg. Geom. Topology, 2009 to appear.

[MSS02] M. Markl, S. Shnider, J.Stasheff. Operads in algebra, topology and physics. Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002.

[M02] G. Moore. Lectures on branes, K-theory and RR charges, Isaac Newton Institute, Cambridge, UK, 2002. (http://www.physics.rutgers.edu/~gmoore/clay.html.)

4 TQFTs from the Kauffman bracket and Integral TQFTs, by Gregor Masbaum

According to Atiyah and Segal's axioms, a Topological Quantum Field Theory (TQFT) describes how to compute quantum invariants of 3-dimensional manifolds by cutting and pasting. This is the point of view we will take in this lecture series. In the first three lectures, I plan to explain the construction of TQFTs from the Kauffman bracket following [BHMV]. This is a very concrete approach to the Reshetikhin–Turaev $SU(2)$- and $SO(3)$-TQFTs which avoids quantum groups. (For the general story, see for example Turaev's book [Tu]. See also Witten [Wi] for the physicist's point of view.) In the last lecture, I will then focus on the $SO(3)$-TQFTs at odd primes which admit a natural integral structure [GM], meaning, for example, that we get representations of surface mapping class groups by matrices with integer coefficients. If time allows, I hope to discuss some applications of this as well.

Prerequisites for this lecture series: In the beginning, only elementary topology and algebra are needed. It will be helpful if students know a little bit of knot theory, such as how knot diagrams in the plane describe knots in 3-space, and how to compute the linking number of two oriented knots in $\mathbb{R}^3$ (but we will review this at the beginning). Students who want to prepare more should study the construction of the Jones polynomial of knots from the Kauffman bracket. All of this can be found, for example, in [L] or [Ro]. In later lectures, some more topology will be needed, but I plan to explain it as we go along.

A rough plan of the lectures could be as follows.

1. Knots and knot diagrams. Kauffman bracket and Jones polynomial. Skein modules and Jones–Wenzl idempotents. Colored Jones polynomial. Extension to an invariant of colored trivalent graphs following [MV].

2. Kirby's theorem (how to describe 3-manifolds by knots and links.) Reshetikhin–Turaev invariant of closed 3-manifolds from the Kauffman bracket. Naïve definition of TQFT.

3. Brief discussion of signatures and Maslov indices. Extended cobordism category. Definition of the TQFT vector space $V_p(\Sigma)$ associated to a closed surface Σ and elementary TQFT-properties. Statement of main theorems: finite dimensionality, tensor product axiom [BHMV]. Some ideas of proofs.

4. Integral TQFT following [GM]: Some elementary number theory in the cyclotomic field $\mathbb{Q}(\zeta_p)$. Definition of the integral lattice $\mathcal{S}_p(\Sigma)$ inside $V_p(\Sigma)$. Main properties and some applications of Integral TQFT.

A few selected references for this lecture series:

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel. Topological Quantum Field Theories derived from the Kauffman bracket, *Topology* **34** (1995), 883–927.

[GM] P. Gilmer, G. Masbaum. Integral lattices in TQFT. *Ann. Sci. École Norm. Sup.* **40** (2007) 815–844.

[L] W. B. R. Lickorish. *An Introduction to knot theory.* Springer GTM 175.

[MV] G. Masbaum, P. Vogel. 3-valent graphs and the Kauffman bracket. *Pacific J. Math.* **164** (1994) 361–381.

[Ro] J. D. Roberts. *Knots Knotes.* Unpublished lecture notes, available at `http://math.ucsd.edu/~justin/papers.html`.

[Tu] V. G. Turaev. *Quantum Invariants of Knots and* 3*-Manifolds.* de Gruyter (1994).

[Wi] E. Witten. Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989) 351–399.