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Ends and Limits of a Dynamical System

J. Manuel García Calcines
L. Javier Hernández Paricio
M. Teresa Rivas Rodríguez

Departamento de Matemática Fundamental
Universidad de La Laguna

Departamento de Matemáticas y Computación
Universidad de La Rioja

1. Elementary concepts on exterior spaces and dynamical systems

1.1. Proper maps and exterior spaces

A continuous map $f : X \to Y$ between topological spaces is said to be **proper** if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of topological spaces and continuous maps and the subcategory of topological spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera.

An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and exterior maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps.

Definition 1 *Let $(X, \mathbf{t})$ be a topological space, where X is the subjacent set and $\mathbf{t}$ its topology. An externology on $(X, \mathbf{t})$ is a non empty collection ε of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \mathbf{t}$ and $E \subset U$ then $U \in \varepsilon$. If an open subset is a member of ε is said to be an exterior open subset.*
An **exterior space** *$(X, \varepsilon, \mathbf{t})$ consists of a space $(X, \mathbf{t})$ together with an externology ε.*
A map $f : (X, \varepsilon, \mathbf{t}) \to (X', \varepsilon', \mathbf{t}')$ is said to be an **exterior map** *if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

An important externology is the family $\varepsilon^{\mathbf{c}}(X)$ of the complements of closed-compact subsets of X , that will be called the cocompact externology.
The new category of exterior spaces and exterior maps, **E**, is complete and cocomplete and contains **P** as a full subcategory via the full embedding

$$(\cdot)^{\mathbf{c}} : \mathbf{P} \hookrightarrow \mathbf{E}\ .$$

The functor $(\cdot)^{\mathbf{c}}$ carries a topological space X to the exterior space $X^{\mathbf{c}}$ which is provided with the topology of X and the externology $\varepsilon^{\mathbf{c}}(X)$.
We also consider the functor

$$(\cdot)\bar{\times}(\cdot)\colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}$$

given by the following construction:
Let $(X, \varepsilon^X, \mathbf{t}_X)$ be an exterior space, $(Y, \mathbf{t}_Y)$ a topological space and for $y \in Y$ we denote by $(\mathbf{t}_Y)_y$ the family of open neighborhoods of Y at y. We consider on $X \times Y$ the product topology $\mathbf{t}_{X\times Y}$ and the externology $\varepsilon^{X\bar{\times}Y}$ given by those $E \in \mathbf{t}_{X\times Y}$ such that for each $y \in Y$ there exists $U_y \in (\mathbf{t}_Y)_y$ and $T^y \in \varepsilon^X$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X\bar{\times}Y$ in order to avoid a possible confusion with the product externology.

1.2. Dynamical Systems and Ω-Limits

Next we recall some basic notions about dynamical systems.

Definition 2 *A* **dynamical system** *(or* **flow***) on a topological space X is a continuous map $\varphi\colon \mathbb{R}\times X \to X$ such that*
(i) $\varphi(0, p) = p, \quad \forall p \in X$
(ii) $\varphi(t, \varphi(s, p)) = \varphi(t + s, p), \quad \forall p \in X, \forall t, s \in \mathbb{R}$. A flow on X will be denoted by (X, φ) and when no confusion be possible, we use X and $t \cdot x = \varphi(t, x)$ for short.
Given two flows (X, ϕ), (Y, ψ), a **flow morphism** *$f\colon (X, \phi) \to (Y, \psi)$ is a continuous map $f\colon X \to Y$ such that $f(r \cdot p) = r \cdot f(p)$ for every $r \in \mathbb{R}$ and for every $p \in X$.*
A subset S of a flow X is said to be **invariant** *if for every $p \in S$ and every $t \in \mathbb{R}$, $t \cdot p \in S$.*

We denote by **F** the category of flows and flows morphisms.

Definition 3 *Let X be a flow. The $\omega^{\mathbf{r}}$***-limit** *set of a point $p \in X$ is given by*

$$\omega^{\mathbf{r}}(p) = \{q \in X | \exists t_n \to +\infty \text{ such that } t_n \cdot p \to q\}$$

*and the Ω^{r}***-limit** *of X by*

$$\Omega^{\mathbf{r}}(X) = \bigcup_{p\in X} \omega^{\mathbf{r}}(p).$$

Definition 4 *Let X be a flow and x a point of X.*

(i) x is a **critical** *point (or a rest point, or an equilibrium point) if for every $r \in \mathbb{R}$, $r \cdot x = x$,*

(ii) x is a **periodic** *point if there is $r \in \mathbb{R}$, $r \neq 0$, such that $r \cdot x = x$,*

(iii) x is **r-Poison stable** *if there is a divergent sequence $t_n \to +\infty$ such that $t_n \cdot x \to x$.*

We denote by $C(X)$, $P(X)$, $P^{\mathbf{r}}(X)$ the invariant subsets of critical, periodic and **r**-Poison stable points of X, respectively. Then,

$$C(X) \subset P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X)\ .$$

The notions above can be dualized to obtain the notion of the $\omega^{\mathbf{l}}$-limit set of a point p, the $\Omega^{\mathbf{l}}$- limit of X and **l**-Poison stable points.

2. End and Limit space functors for the category of exterior spaces

For a topological space Y, $\pi_0(Y)$ denotes the set of path-components of Y and we have a continuous canonical map $Y \to \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$.

Definition 5 *Given an exterior space $X = (X, \varepsilon(X))$, the topological subspace:*

$$L(X) = \lim \varepsilon(X) = \cap_{E\in\varepsilon(X)} E$$

will be called the **limit space** *of X .*
The **end space** *of X is the inverse limit:*

$$\check{\pi}_0(X) = \lim \pi_0\varepsilon(X) = \lim_{E\in\varepsilon(X)} \pi_0(E)$$

provided with the inverse limit topology of the spaces $\pi_0(E)$.

Note that an end point $a \in \check{\pi}_0(X)$ is represented by the filter base:

$$\{U^E_a | U^E_a \text{ is a path-component of } E, E \in \varepsilon(X)\}\ .$$

On the other hand, when X is locally path-connected, then we have that $\check{\pi}_0(X)$ is a prodiscrete space.

Given an exterior space $X = (X, \varepsilon(X))$ one has a canonical continuous map

$$e\colon L(X) \to \check{\pi}_0(X).$$

This permits to decompose the limit of an exterior space:

Definition 6 *Given an exterior space X, an end point $a \in \check{\pi}_0(X)$ is said to be* **representable** *by $b \in L(X)$ if $e(b) = a$. Notice that the map $e\colon L(X) \to \check{\pi}_0(X)$ induce an e***-decomposition**

$$L(X) = \bigsqcup_{a\in\check{\pi}_0(X)} L_a(X)$$

*where $L_a(X) = e^{-1}(a)$ will be called the a***-component** *of the limit $L(X)$.*

Concerning this e-decomposition of the limit there are some interesting questions that have to be studied; for instance, under which conditions one has that $L(X)$ or $L_a(X)$ are compact spaces. It will also be interesting to analyze the exterior spaces whose limit components $L_a(X)$ are continua.
If X, Y are exterior spaces and $f\colon X \to Y$ is an exterior map, then f induces continuous maps $L(f)\colon L(X) \to L(Y)$, $\check{\pi}_0(f)\colon \check{\pi}_0(X) \to \check{\pi}_0(Y)$ and we have the functors:

$$L, \check{\pi}_0\colon \mathbf{E} \to \mathbf{Top}.$$

It is not difficult to check that L preserves exterior homotopies and $\check{\pi}_0$ is invariant by exterior homotopy:

Lemma 1 *Suppose that X, Y be exterior spaces and $f, g\colon X \to Y$ are exterior maps.*

(i) If $H\colon X\bar{\times}I \to Y$ is an exterior homotopy from f to g, then $L(H) = H|_{L(X)\times I}\colon L(X\bar{\times}I) = L(X) \times I \to L(Y)$ is a homotopy from $L(f)$ to $L(g)$.

(ii) If $H\colon X\bar{\times}I \to Y$ is an exterior homotopy from f to g, then $\check{\pi}_0(f) = \check{\pi}_0(g)$.

Then, if $\pi\mathbf{E}$ and $\pi\mathbf{Top}$ are the exterior homotopy category and the usual homotopy category corresponding to **E** and **Top** respectively, one has the following result.

Proposition 1 *The functors $L\colon \mathbf{E} \to \mathbf{Top}$, $\check{\pi}_0\colon \mathbf{E} \to \mathbf{Top}$ induce functors*

$$L\colon \pi\mathbf{E} \to \pi\mathbf{Top}\,, \quad \check{\pi}_0\colon \pi\mathbf{E} \to \mathbf{Top}\ .$$

3. Exterior Flows. End and Limit spaces of a dynamical system via exterior spaces

We consider the following externology on $\mathbb{R}$:

$$\mathbf{r} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } [n, +\infty) \subset U\}$$

and we denote the corresponding exterior space by $\mathbb{R}^{\mathbf{r}}$. Note that a base for $\mathbf{r}$ is given by

$$\mathcal{B}(\mathbf{r}) = \{[n, +\infty) | n \in \mathbb{N}\}\ .$$

We propose the following notion that mixes the structures of dynamical system and exterior space:

Definition 7 *Let M be an exterior space, $M_{\mathbf{t}}$ the subjacent topological space and $M_{\mathbf{d}}$ the set M provided with the discrete topology. An* **r-exterior flow** *is a continuous flow $\phi\colon \mathbb{R}\times M_{\mathbf{t}} \to M_{\mathbf{t}}$ such that $\phi\colon \mathbb{R}^{\mathbf{r}}\bar{\times}M_{\mathbf{d}} \to M$ is exterior and for any $s \in \mathbb{R}$, $\phi_s\colon M \to M$ is also exterior.*
An **r-exterior flow morphism** *of $\mathbf{r}$-exterior flows $f\colon M \to N$ is a flow morphism such that f is exterior.*

Denote by $\mathbf{E^rF}$ the category of **r**-exterior flows and **r**-exterior flow morphisms.

We have defined above the end and limit space of an exterior space. In particular, since an **r**-exterior flow X is an exterior space, we can consider the end space $\check{\pi}_0(X)$ and the limit space $L(X)$. Then we have:

Proposition 2 *Suppose that (X, ϕ) is an $\mathbf{r}$-exterior flow. Then,*

(i) the space $L(X)$ is invariant,

(ii) there is a trivial induced flow on $\check{\pi}_0(X)$.

We note that for an **r**-exterior flow X, each trajectory has an end point given as follows:
If $p \in X$ and $E \in \varepsilon(X)$, there is $T^p \in \mathbf{r}$ such that $T^p \cdot p \subset E$. We can suppose that T^p is path-connected, then $T^p \cdot p$ is path-connected and there is a unique $\omega_{\mathbf{r}}(p, E)$ path-component of E such that $T^p \cdot p \subset \omega_{\mathbf{r}}(p, E) \subset E$. This gives maps $\omega_{\mathbf{r}}(\cdot, E)\colon X \to \pi_0(E)$ and $\omega_{\mathbf{r}}\colon X \to \check{\pi}_0(X)$ such that the following diagram commutes:

$$\begin{array}{ccc} L(X) & & \\ \downarrow & \overset{e}{\searrow} & \\ X & \xrightarrow{\omega_{\mathbf{r}}} & \check{\pi}_0(X) \end{array}$$

The map $\omega_{\mathbf{r}}$ permits to divide an **r**-exterior flow in simpler **r**-exterior flows:

Definition 8 *Let X be an $\mathbf{r}$-exterior flow. The invariant space denoted by*

$$X_{(\mathbf{r},a)} = \omega_{\mathbf{r}}^{-1}(a), \quad a \in \check{\pi}_0(X)$$

will be called the **r-basin** *at a.*
The induced partition of X in simpler $\mathbf{r}$-exterior flows:

$$X = \bigsqcup_{a\in\check{\pi}_0(X)} X_{(\mathbf{r},a)}$$

will be called the $\omega_{\mathbf{r}}$**-decomposition** *of the $\mathbf{r}$-exterior flow X.*

Given an **r**-exterior flow $(M, \phi) \in \mathbf{E^rF}$, one also have a flow $(M_{\mathbf{t}}, \phi) \in \mathbf{F}$. This gives a forgetful functor

$$(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF} \to \mathbf{F}\ .$$

Now given a flow (X, φ), an open $N \in \mathbf{t}_X$ is said to be **r-exterior** (or absorbing open) if for any $x \in X$ there is $T^x \in \mathbf{r}$ such that $\varphi(T^x \times \{x\}) \subset N$.
It is easy to check that the family of **r**-exterior subsets of X is an externology, that will be denoted by $\varepsilon^{\mathbf{r}}(X)$, which gives an exterior space $X^{\mathbf{r}}$ and $\varphi\colon \mathbb{R}^{\mathbf{r}}\bar{\times}X_{\mathbf{d}} \to X^{\mathbf{r}}$ is an **r**-exterior flow. The pair $(X^{\mathbf{r}}, \varphi)$ is said to be the **r**-exterior flow associated to X . When there is no possibility of confusion, $(X^{\mathbf{r}}, \varphi)$ will be briefly denoted by $X^{\mathbf{r}}$. Then we have a functor

$$(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{E^rF}.$$

The forgetful functor and the given construction of exterior flows are related as follows:

Proposition 3 *The functor $(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{E^rF}$ is left adjoint to the functor $(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF} \to \mathbf{F}$. Moreover $(\cdot)_{\mathbf{t}}\,(\cdot)^{\mathbf{r}} = \mathrm{id}$ and $\mathbf{F}$ can be considered as a full subcategory of $\mathbf{E^rF}$ via $(\cdot)^{\mathbf{r}}$.*

Definition 9 *Given a flow X, the space $\check{\pi}^{\mathbf{r}}_0(X) = \check{\pi}_0(X^{\mathbf{r}})$ is said to be the* **end space of the flow** *X and the space $L^{\mathbf{r}}(X) = L(X^{\mathbf{r}})$ is said to be the* **limit space of the flow** *X .*

We remark that if X is a flow their associated **r**-exterior structure permits to decompose X using the decomposition of $X^{\mathbf{r}}$. The $\omega_{\mathbf{r}}$-decomposition

$$X^{\mathbf{r}} = \bigsqcup_{a\in\check{\pi}^{\mathbf{r}}_0(X)} X^{\mathbf{r}}_{(\mathbf{r},a)}$$

can be considered as generalization for a continuous flow of a disjoint union of "stable"submanifolds of a differentiable flow . On the other side, the above decomposition generalizes Morse-Smale's decompositions of dynamical system associated to Morse functions .
It is interesting to note that the $\omega_{\mathbf{r}}$-decomposition of X is compatible with the e-decomposition of the limit subspace $L(X)$.
The relation of the limit space of a flow or an **r**-exterior flow and the subflow of periodic points is analysed in the following results:

Lemma 2 *Let X be an $\mathbf{r}$-exterior flow and suppose that $x \in X$. If x is a periodic point, then for every $E \in \varepsilon(X)$, $x \in E$.*

Proposition 4 *Let X be an $\mathbf{r}$-exterior flow. Then, $P(X) \subset L(X)$.*

Lemma 3 *Let X be a flow and suppose that X is a T_1-space. Then, for every $x \in X$ the following statements are equivalent:*

(i) x is a non-periodic point,

(ii) $X \setminus \{x\}$ is an $\mathbf{r}$-exterior subset of X.

Theorem 1 *Let X be a flow and suppose that X is a T_1-space. Then, $L^{\mathbf{r}}(X) = P(X)$ the set of periodic points of X .*

Taking into account the result above, if X is flow and X is T_1 we have that

$$L^{\mathbf{r}}(X) = P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) \subset X\ .$$

With respect to decompositions, it will be interesting to find topological and dynamical conditions to ensure that the $\omega_{\mathbf{r}}$-decomposition of a flow X divides $\Omega^{\mathbf{r}}(X)$ without dividing $\omega^{\mathbf{r}}(x)$ for each $x \in X$.

We note that if we take on $\mathbb{R}$ the externology

$$\mathbf{l} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (-\infty, -n] \subset U\}$$

or we take the reversed flow, we have the notion of **l**-exterior flow and we obtain the corresponding dual results.