Fuzzy metrics and color image filtering

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Abstract

Occasionally, advances in the properties of fuzzy metrics are hindered because only a few examples of fuzzy metrics are known and so their application in Engineering methods is limited. To overcome both inconveniences, in this paper we provide some new examples of fuzzy metrics. Also we show how to apply fuzzy metrics in engineering methods. In this sense we evaluate the proximity of two pixels of a color image by means of a fuzzy metric which combines two fuzzy metrics associated to two distinct proximity criteria, respectively. As a result, the image processing filter built works better than other classical ones.

1. Introduction

The interest of fuzzy metrics is mainly due to the following two main advantages with respect to classical metrics: First, values given by fuzzy metrics are in the interval]0,1] regardless the nature of the distance concept being measured. This implies that it is easy to combine different distance criteria that may originally be in quite different ranges but fuzzy metrics take to a common range. In this way, the combination of several distance criteria may be done in a straightforward way. Second, fuzzy metrics match perfectly with the employment of other fuzzy techniques since the value given by a fuzzy metric can be directly employed or interpreted as a fuzzy certainty degree. This allows to straightforwardly include fuzzy metrics as part of other complex fuzzy systems.

2. Preliminaries

Definition 1 *([2]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, +\infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:*

- (GV1) $M(x, y, t) > 0;$
- (GV2) $M(x, y, t) = 1$ *if and only if* $x = y;$
- (GV3) $M(x, y, t) = M(y, x, t);$
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s);$
- (GV5) $M(x, y, \_) :]0, +\infty[\rightarrow]0, 1]$ *is continuous.*

Definition 2 *A fuzzy metric M on X is said to be stationary, [3], if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant.*

3. Examples of fuzzy metrics

Example 1 *Let $f : X \to \mathbb{R}^+$ be a one-to-one function and let $g : \mathbb{R}^+ \to [0, +\infty[$ be an increasing continuous function. Fixed $\alpha, \beta > 0$, define M by*

$$M(x, y, t) = \left(\frac{(\min\{f(x), f(y)\})^\alpha + g(t)}{(\max\{f(x), f(y)\})^\alpha + g(t)} \right)^\beta \qquad (1)$$

Then, $(M, \cdot)$ is a fm *on X.*

Now, if we take f as the corresponding identity function and $\alpha = \beta = 1$ then we obtain the next three examples as particular cases.

(A) Let $X = \mathbb{R}^+$, and let g be the identity function. Then (1) becomes

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

and this fm was given in [6], Example 2.5.

(B) Let $X = \mathbb{N}$ and take g as the zero function. Then (1) becomes

$$M(x, y) = \frac{\min\{x, y\}}{\max\{x, y\}} \qquad (2)$$

and this sfm was given in [2], Example 2.11. The same assertion is true if $X = \mathbb{R}^+$.

(C) Let $X =]-k, +\infty[$ $(k > 0)$, and take g as the constant function $g(t) = k$. Then, (1) becomes

$$M(x, y) = \frac{\min\{x, y\} + k}{\max\{x, y\} + k} \qquad (3)$$

and this sfm was used in [4]).

It is easy to verify that, in general, $(M, \wedge)$ is not a fm. In the next two examples $g : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function, and d is a metric on X. The open balls centered at x with radius $R > 0$ in (X, d) will be denoted by $B_R(x)$ and the topology on X deduced from d will be denoted by $\tau(d)$.

Example 2 *Let $m > 0$. Define the function M by*

$$M(x, y, t) = \frac{g(t)}{g(t) + m \cdot d(x, y)} \qquad (4)$$

Then $(M, \cdot)$ is a fm *on X.*

Notice that in this case for each $x \in X$, $t > 0$ and $r \in]0, 1[$ we have $B(x, r, t) = B_R(x)$, where $R = \frac{g(t)}{m} \cdot \frac{r}{1 - r}$.
On the other hand, for each $x \in X$ and $R > 0$ we have that $B_R(x) = B(x, r, t)$ where $r = 1 - \frac{g(t)}{g(t) + mR}$ for each $t > 0$ and so τ_M agrees with $\tau(d)$.
As a particular case if we take $g(t) = t^n$ where $n \in \mathbb{N}$ and $m = 1$, then (4) becomes

$$M(x, y, t) = \frac{t^n}{t^n + d(x, y)} \qquad (5)$$

and so $(M, \wedge)$ is a fm as it was shown in [5]. In particular, for $n = 1$ the well-known *standard fuzzy metric*, defined in [2], is obtained.
On the other hand, if we take in equation (4) g as a constant function, $g(t) = k > 0$, and $m = 1$, we obtain

$$M(x, y, t) = \frac{k}{k + d(x, y)}$$

and so $(M, \cdot)$ is a sfm on X but, in general, $(M, \wedge)$ is not.

Example 3 *Define the function M by*

$$M(x, y, t) = e^{-\frac{d(x,y)}{g(t)}} \qquad (6)$$

Then $(M, \cdot)$ is a fm *on X.*

As a particular case, if we take g as the identity function, then (6) becomes

$$M(x, y, t) = e^{-\frac{d(x,y)}{t}}$$

In this case $(M, \wedge)$ is a fm on X which can be found in [2], Remark 2.8.
On the other hand if we take g as a constant function $g(t) = k > 0$, then (6) becomes

$$M(x, y, t) = e^{-\frac{d(x,y)}{k}}$$

and so $(M, \cdot)$ is a fm but, in general, $(M, \wedge)$ is not.

Example 4 *Let (X, d) be a bounded metric space and suppose $d(x, y) < k$ for all $x, y \in X$. Let $g : \mathbb{R}^+ \to]k, +\infty[$ be an increasing continuous function. Define the function M by*

$$M(x, y, t) = 1 - \frac{d(x, y)}{g(t)} \qquad (7)$$

Then $(M, \mathfrak{L})$ is a fm *on X.*

If we take g as a constant function $g(t) = K > k$, then (7) becomes

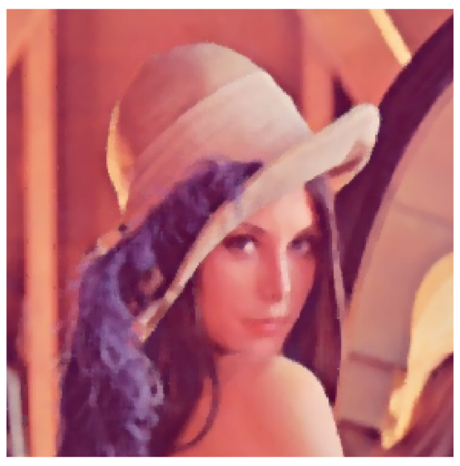
$$M(x, y) = 1 - \frac{d(x, y)}{K}$$

and so $(M, \mathfrak{L})$ is a sfm but, in general, $(M, \cdot)$ is not.

4. Image filtering using fuzzy metrics

The *Vector Median Filter* (VMF) [1] employs the Euclidean metric as the distance criteria between the vectors. This filter is known to behave very robustly but the resulting images are frequently too smoothed, and so, edges and details are not properly preserved. In the approach we propose, the inclusion of the spatial criterion helps to improve the preservation of image details while the noise is also reduced. This is achieved because the output vector in each filtering window is determined as a vector which is spatially close to all the other vectors in the window. Therefore, we avoid the possibility of replacing a pixel with another located far from it, which is not appropriate to preserve edges and details. Figure 1 shows three noisy images filtered with both the VMF and the proposed method. We can see that both methods are able to suppress the noise but, whereas the VMF generates too blurry output images, the proposed method is able to better preserve image edges and details, and so, to improve the visual quality of the obtained images.

(a)



(b)

(c)

Figure 1:
(a) Detail of Lenna image with 10% of noise, (b) output using the VMF with a filter window of size 5×5, (c) output of the proposed method.

References

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