Evaluating Interval-Valued Influence Diagrams

PGMs for scalable data analytics meeting
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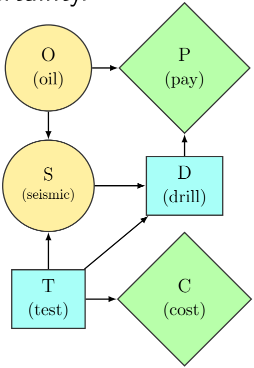
R. Cabañas, A. Antonucci, A. Cano, and M Gómez-Olmedo. "Variable Elimination for Interval-Valued Influence Diagrams". In: *Symbolic and Quantitative Approaches to Reasoning with Uncertainty: 13th European Conference, ECSQARU 2015, Compiègne, France, July 15-17, 2015. Proceedings*. Vol. 9161 LNAI. Springer, 2015, pp. 541–551

R. Cabañas, A. Antonucci, A. Cano, and M Gómez-Olmedo. "Evaluating Interval-Valued Influence Diagrams". Submitted to International Journal of Approximate Reasoning, Special Issue ECSQARU 2015

Introduction

Influence Diagrams (Precise)

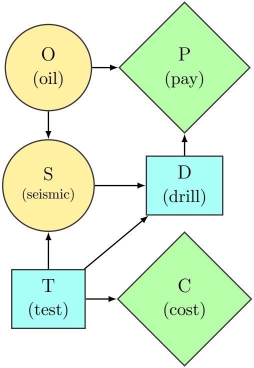
- Influence diagrams (IDs) are PGMs used to solve decision problems under uncertainty.



Oil wildcatter ID (Raiffa 1968, Shenoy 1992) - `http://www.hugin.com/`

Influence Diagrams (Precise)

- 3 types of nodes: *chance*, *decision* and *utility*



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Influence Diagrams (Precise)

- probability and utility potentials (sharp numerical values)

$\phi(O)$

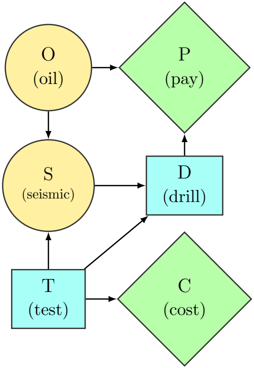
O: e	w	s
.5	.3	.2

$\phi(S|O, T = t)$

		O: e	w	s
S	c	.1	.3	.5
	o	.3	.4	.4
	d	.6	.3	.1

$\phi(S|O, T = nt)$

		O: e	w	s
S	c	.33	.33	.33
	o	.33	.33	.33
	d	.33	.33	.33



$\psi(O, D)$

		O: e	w	s
T	d	-70	50	200
	nd	0	0	0

$\psi(T)$

T: t	nt
-10	0

Oil wildcatter ID (Raiffa 1968, Shenoy 1992) - `http://www.hugin.com/`

Interval-valued Influence Diagrams

- We aim to extend IDs to support interval-valued specifications.

$\underline{\overline{\phi}}(O)$

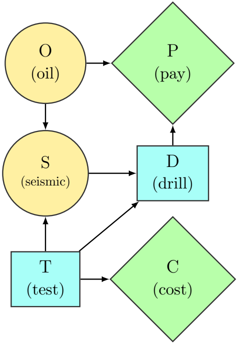
O: e	O: w	O: s
[.4875, .5125]	[.2925, .3175]	[.195, .22]

$\underline{\overline{\phi}}(S|O, T = t)$

		O: e	O: w	O: s
S	c	[.0975, .1225]	[.2925, .3175]	[.4875, .5125]
	o	[.2925, .3175]	[.39, .415]	[.39, .415]
	d	[.585, .61]	[.2925, .3175]	[.0975, .1225]

$\underline{\overline{\phi}}(S|O, T = nt)$

		O: e	O: w	O: s
S	c	[.325, .35]	[.325, .35]	[.325, .35]
	o	[.325, .35]	[.325, .35]	[.325, .35]
	d	[.325, .35]	[.325, .35]	[.325, .35]



$\underline{\overline{\psi}}(O, D)$

		O: e	O: w	O: s
T	d	[−75, −65]	[45, 55]	[195, 205]
	nd	[−5, 5]	[−5, 5]	[−5, 5]

$\underline{\overline{\psi}}(T)$

T: t	T: nt
[−15, −5]	[−5, 5]

Interval-valued Influence Diagrams

- Partially reliable or incomplete **data streams**
- Qualitative expert knowledge

$\underline{\overline{\phi}}(O)$

O: e	w	s
$[.4875, .5125]$	$[.2925, .3175]$	$[.195, .22]$

$\underline{\overline{\phi}}(S|O, T = t)$

		O: e	w	s
S	c	$[.0975, .1225]$	$[.2925, .3175]$	$[.4875, .5125]$
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S	c	$[.325, .35]$	$[.325, .35]$	$[.325, .35]$
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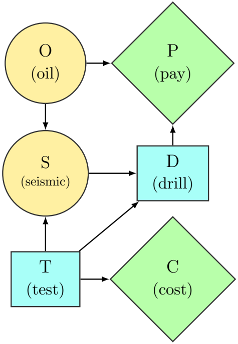

$\underline{\overline{\psi}}(O, D)$

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T	d	$[-75, -65]$	$[45, 55]$	$[195, 205]$
	nd	$[-5, 5]$	$[-5, 5]$	$[-5, 5]$

$\underline{\overline{\psi}}(T)$

T: t	nt
$[-15, -5]$	$[-5, 5]$

Evaluation (inference)

Evaluating an ID implies computing:

- The optimal policy $\delta^*_{D_i} : \Omega_{\Pi_{D_i}} \rightarrow \Omega_{D_i}$ for each decision D_i:
- The expected utility of the optimal strategy $\Delta^* = \{\delta^*_{D_1}, \ldots, \delta^*_{D_n}\}$

Precise Evaluation

	δ^*_D	c (Seismic)	o (Seismic)	d (Seismic)
$Test$	t	d	d	nd
	nt	d	d	d

$$EU(\Delta^*) = 22.5$$

Imprecise Evaluation

	δ^*_D	c (Seismic)	o (Seismic)	d (Seismic)
$Test$	t	d	d	nd
	nt	d	$\{d, nd\}$	$\{d, nd\}$

$$EU(\Delta^*) = [20.0, 25.0]$$

Evaluation (inference) - Variable Elimination

Chance and decision nodes are removed in reverse order:

$\Phi = \{\phi(O), \phi(S|O,T)\}$ $\Psi = \{\psi(O,D), \psi(T)\}$

Removal of O (chance):

1. Select and combine relevant potentials (with variable O):
   $\phi(O,S|T) \leftarrow \phi(O) \otimes \phi(S|O,T)$ (probabilities)
   $\psi(O,D)$ (utilities)
2. Remove by sum:
   $\phi(S|T) \leftarrow \sum_O \phi(O,S|T)$ $\psi(S,T,D) \leftarrow \frac{\sum_O \phi(O,S|T) \otimes \psi(O,D)}{\sum_O \phi(O,S|T)}$

Update potential sets:

$\Phi = \{\phi(S|T)\}$ $\Psi = \{\psi(S,T,D), \psi(T)\}$

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Evaluation (inference) - Variable Elimination

$\Phi = \{\phi(S|T)\}$ $\Psi = \{\psi(S,T,D), \psi(T)\}$

Removal of D (decision):

1. Select and combine relevant potentials (with variable D):
   $\psi(S,T,D)$ (utilities)
2. Remove by max:
   $\psi(S,T) \leftarrow \max_D \psi(S,T,D)$
   $\delta^*_D(S,T) \leftarrow \arg\max_D \delta^*_D \psi(S,T,D)$

Update potential sets:
$\Phi = \{\phi(S|T)\}$ $\Psi = \{\psi(S,T), \psi(T)\}$

This process is repeated with all the variables

Evaluation (inference) - Variable Elimination

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This process is repeated with all the variables

Interval-Valued Potentials

- What is an **interval-valued potential**?
- Sensitivity-analysis interpretation:

an interval-valued potential is a collection of precise potentials consistent with the interval constraints

Interval-Valued Utility Potentials (IUPs)

An **interval utility potential** $\underline{\overline{\psi}}(X_I)$ is a pair of precise utility potentials over X_I:

- $\underline{\psi}(X_I), \overline{\psi}(X_I)$ are the lower and upper bounds.
- The *extension* $\underline{\overline{\psi}}^*(X_I)$ is the set of UPs consistent with the bounds, i.e.,

$$\underline{\overline{\psi}}^*(X_I) := \left\{ \psi : \Omega_{X_I} \to \mathbb{R} \,\middle|\, \underline{\psi}(x_I) \leq \psi(x_I) \leq \overline{\psi}(x_I), \forall x_I \in \Omega_{X_I} \right\}$$

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$\psi(O, D)$

		O: e	w	s
T	d	$[-75, -65]$	$[45, 55]$	$[195, 205]$
	nd	$[-5, 5]$	$[-5, 5]$	$[-5, 5]$

Interval-Valued Probability Potentials (IPPs)

An **interval probability potential** $\underline{\overline{\phi}}(X_I|X_J)$ is a pair of precise probability potentials over $X_I \cup X_J$:

- $\underline{\phi}(X_I|X_J), \overline{\phi}(X_I|X_J)$ are the lower and upper bounds.
- The *extension* $\underline{\overline{\phi}}^*(X_I|X_J)$ is the set of PPs consistent with the bounds, i.e.,

$$\underline{\overline{\phi}}^*(X) := \left\{ \phi : \Omega_{X_I} \times \Omega_{X_J} \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{l} \sum_{x_I} \phi(x_I|x_J) = 1, \\ \underline{\phi}(x_I|x_J) \leq \phi(x_I|x_J) \leq \overline{\phi}(x_I|x_J), \\ \forall (x_I, x_J) \in \Omega_{X_I} \times \Omega_{X_J} \end{array} \right\}.$$

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$\phi(S|O, T = t)$

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S	c	$[.0975, .1225]$	$[.2925, .3175]$	$[.4875, .5125]$
	o	$[.2925, .3175]$	$[.39, .415]$	$[.39, .415]$
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$\phi(S|O, T = nt)$

		O: e	w	s
S	c	$[.325, .35]$	$[.325, .35]$	$[.325, .35]$
	o	$[.325, .35]$	$[.325, .35]$	$[.325, .35]$
	d	$[.325, .35]$	$[.325, .35]$	$[.325, .35]$

Evaluation of Interval-Valued Influence Diagrams

Combination of Interval-Valued Potentials

The combination (multiplication or addition) can be performed by operating with the lower and upper bounds of each potential:

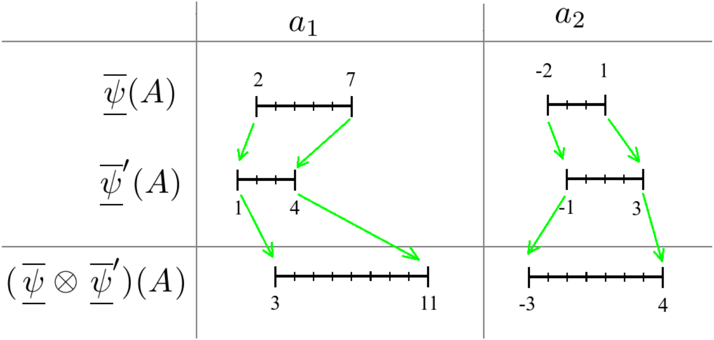
i) Two IUPs $\underline{\overline{\psi}}(X_I)$ and $\underline{\overline{\psi}}'(X_J)$:

$$(\overline{\psi} \otimes \overline{\psi}')(x_{I\cup J}) := \overline{\psi}(x_I) + \overline{\psi}'(x_J)$$
$$(\underline{\psi} \otimes \underline{\psi}')(x_{I\cup J}) := \underline{\psi}(x_I) + \underline{\psi}'(x_J) \qquad \forall x_{I\cup J} \in \Omega_{X_{I\cup J}}$$

ii) Two IPPs $\underline{\overline{\phi}}(X_I|X_J)$ and $\underline{\overline{\phi}}'(X_K|X_L)$:

$$(\overline{\phi} \otimes \overline{\phi}')(x_{I\cup K}|x_{(J\cup L)\setminus(I\cup K)}) := \overline{\phi}(x_I|x_J) \cdot \overline{\phi}(x_K|x_L)$$
$$(\underline{\phi} \otimes \underline{\phi}')(x_{I\cup K}|x_{(J\cup L)\setminus(I\cup K)}) := \underline{\phi}(x_I|x_J) \cdot \underline{\phi}(x_K|x_L)$$
$$\forall\, x_{I\cup K} \in \Omega_{X_{I\cup K}},\;\; x_{(J\cup L)\setminus(I\cup K)} \in \Omega_{X_{(J\cup L)\setminus(I\cup K)}}$$

Combination of Interval-Valued Potentials



Combination of Interval-Valued Potentials

iii) An IPP $\underline{\overline{\phi}}(X_I|X_J)$ and an IUP $\underline{\overline{\psi}}(X_K)$

$$(\overline{\phi} \otimes \overline{\psi})(x_{I\cup J\cup K}) := \left\{ \begin{array}{ll} \overline{\phi}(x_I|x_J) \cdot \overline{\psi}(x_K) & if \;\; \overline{\psi}(x_K) > 0 \\ \underline{\phi}(x_I|x_J) \cdot \overline{\psi}(x_K) & otherwise \end{array} \right\}$$

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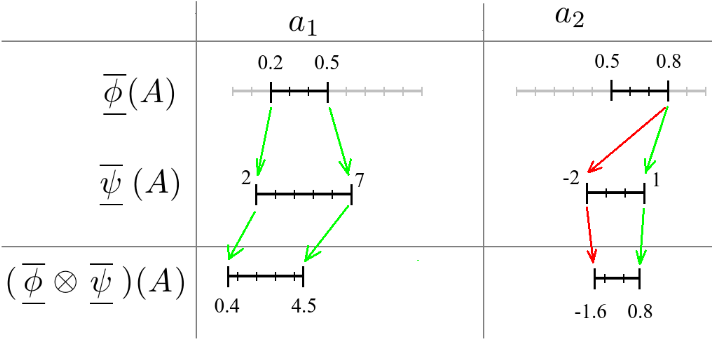
$$\forall x_{I\cup J\cup K} \in \Omega_{X_{I\cup J\cup K}}$$

Combination of Interval-Valued Potentials



VE for evaluating IIDs

How can we adapt VE for IIDs? We consider 3 alternatives:

1. Defining an algebra with interval-valued potentials (combination, division, marginalization, etc.)
   - Each operation is defined element-wise by operating independently with the bounds:

$$\frac{[10,20]}{[0.1,0.5]} = \left[\frac{10}{0.5}, \frac{20}{0.1}\right] = [20, 200]$$

   - **Problem:** large intervals
   - Intervals including ∞:

$$\frac{[10,20]}{[0.0,0.5]} = \left[\frac{10}{0.5}, \frac{20}{0.0}\right] = [20, +\infty]$$

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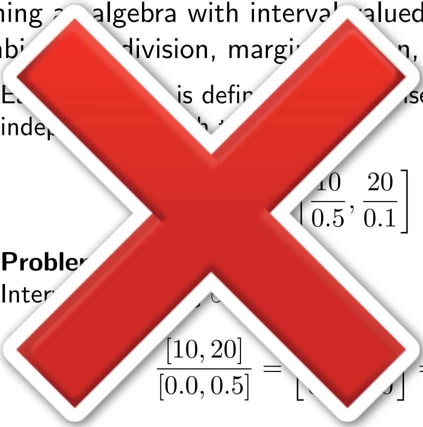
     $$\frac{10}{0.5}, \frac{20}{0.1}\Big] = [20, 200]$$
   - **Probler**
   - Inter

$$\frac{[10, 20]}{[0.0, 0.5]} = \Big[\quad \Big] = [20, +\infty]$$



VE for evaluating IIDs

How can we adapt VE for IIDs? We consider 3 alternatives:

2. Outter Approximation

- Each operation **except the division** is defined element-wise by operating independently with the bounds
- The removal of a chance variable from IPPs is approximated:

$$\overline{\hat{\psi}}(x_I, x_J|x_K) = \sum_{i=1}^{|\Omega_Y|} \left(\frac{\overline{\phi}(y_i, x_I|x_J)}{\overline{\phi}(y_i, x_I|x_J) + \sum_{j \neq i} \underline{\phi}(y_i, x_I|x_J)} \cdot \overline{\psi}(y_i, x_K) \right)$$

$$\underline{\overline{\phi}}(y_1, \ldots) = [0.0, 0.2], \quad \underline{\overline{\phi}}(y_2, \ldots) = [0.0, 0.2], \quad \underline{\overline{\phi}}(y_3, \ldots) = [0.1, 0.25]$$
$$\underline{\overline{\psi}}(y_1, \ldots) = [0, 50], \quad \underline{\overline{\psi}}(y_2, \ldots) = [5, 20], \quad \underline{\overline{\psi}}(y_3, \ldots) = [3, 45]$$

$$\frac{0.2}{0.2+0.0+0.1} \cdot 50 + \frac{0.2}{0.0+0.2+0.1} \cdot 20 + \frac{0.25}{0.0+0.0+0.25} \cdot 45 = 91.66$$

- The lower bound can similarly be computed: 0.6

VE for evaluating IIDs

How can we adapt VE for IIDs? We consider 3 alternatives:

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   - The lower bound can similarly be computed: 0.6

VE for evaluating IIDs

How can we adapt VE for evaluating IIDs? We consider 3 alternatives:

3. Linear Programming:
   - Elimination of a chance variable from IPPs:

$$\sum_{Y} \underline{\overline{\phi}}(X_I|X_J,Y) \otimes \underline{\overline{\phi}}(Y,X_K|X_L)$$

   - Elimination of a chance variable from IUPs:

$$\frac{\sum_{Y} \underline{\overline{\phi}}(X_I|X_J,Y) \otimes \underline{\overline{\phi}}(Y,X_K|X_L) \otimes \underline{\overline{\psi}}(Y,X_M)}{\sum_{Y} \underline{\overline{\phi}}(X_I|X_J,Y) \otimes \underline{\overline{\phi}}(Y,X_K|X_L)}$$

   - Elimination of a decision variable from IUPs:

$$\max_{D} \underline{\overline{\psi}}(D,X_I)$$

VE for evaluating IIDs (Elimination of a chance variable from IPPs)

- Each lower bound $\underline{\phi}(x_{K\cup I}|x_{L\cup J})$ is the solution of:

$$\begin{aligned}
\text{min} \quad & \sum_{y\in\Omega_Y} \phi(x_I|x_J,y)\cdot\phi(y,x_K|x_L), \\
\text{subject to:} \quad & \underline{\phi}(x_I|x_J,y) \leq \phi(x_I|x_J,y) \leq \overline{\phi}(x_I|x_J,y), \\
& \underline{\phi}(y,x_K|x_L) \leq \phi(y,x_K|x_L) \leq \overline{\phi}(y,x_K|x_L), \forall y\in\Omega_Y.
\end{aligned}$$

VE for evaluating IIDs (Elimination of a chance variable from IPPs)

- Each lower bound $\underline{\phi}(x_{K\cup I}|x_{L\cup J})$ is the solution of:

$$\min \quad \sum_{y\in\Omega_Y} \underline{\phi}(x_I|x_J, y) \cdot \phi(y, x_K|x_L),$$

subject to: $\sout{\underline{\phi}(x_I|x_J, y) \leq \phi(x_I|x_J, y) \leq \overline{\phi}(x_I|x_J, y)},$

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- Additional constraints:

$$\sum_y \phi(y,x_K|x_L)\geq 1-\sum_{x'_K\neq x_K}\sum_y \overline{\phi}(y,x'_K|x_L)$$

$$\sum_y \phi(y,x_K|x_L)\leq 1-\sum_{x'_K\neq x_K}\sum_y \underline{\phi}(y,x'_K|x_L)$$

VE for evaluating IIDs (Elimination of a chance variable from IUPs)

- Each lower bound $\underline{\psi}(x_{I \cup J \cup K \cup L \cup M})$ is the solution of:

$$\min \quad \frac{\sum_{y \in \Omega_Y} \phi(x_I|x_J, y) \cdot \phi(y, x_K|x_L) \cdot \psi(y, x_M)}{\sum_{y \in \Omega_Y} \phi(x_I|x_J, y) \cdot \phi(y, x_K|x_L)},$$

subject to:

$$\underline{\phi}(x_I|x_J, y) \leq \phi(x_I|x_J, y) \leq \overline{\phi}(x_I|x_J, y),$$
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$$\sout{\underline{\psi}(y,x_M)\leq\psi(y,x_M)\leq\overline{\psi}(y,x_M)}$$
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$$\phi(y,x_K|x_L)\cdot\phi(x_I|x_J,y)\leq\overline{\phi}(y,x_K|x_L)\cdot\overline{\phi}(x_I|x_J,y)$$

- The task is linear-fractional, so we apply the Charnes Cooper transformation

VE for evaluating IIDs (Elimination of a chance variable from IUPs)

- The removal of a decision D from $\overline{\underline{\psi}}(D, X_I)$ gives as result a new IUP $\overline{\underline{\hat{\psi}}}(X_I)$ s.t.

$$\overline{\underline{\hat{\psi}}}^*(X_I) := \left\{ \psi(X_I) \,\middle|\, \begin{array}{l} \max_D \psi(D, x_I) \\ \forall x_I, \forall \psi \in \overline{\underline{\psi}}^* \end{array} \right\}$$

- The maximization of the IUP is done as by acting separately on the two bounds
- To detect the optimal policy $\delta_D^*(X_I)$ we compute the interval-maximal states of D given each $x_I \in \Omega_{X_I}$ (credal policy)

Empirical Validation

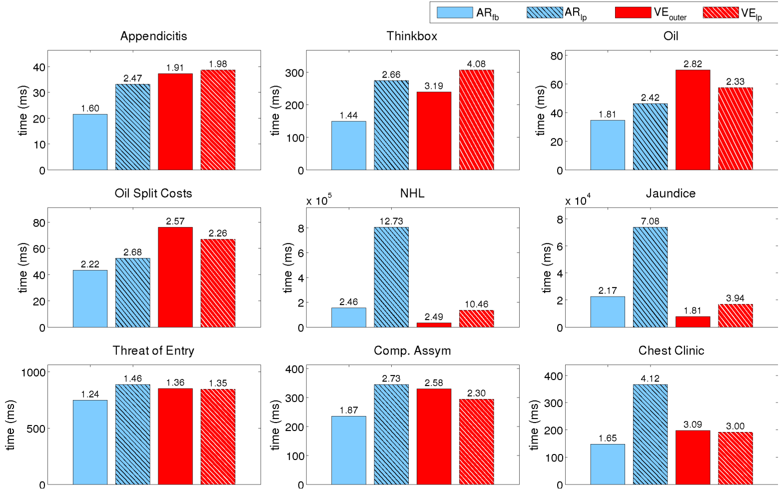
Empirical Validation

- 8 precise IDs were transformed into interval-valued IDs by a perturbation of the original parameters
- 4 different methods are compared:
  1. Arc reversal approach by Feertig and Breese $(\mathrm{AR}_{\mathrm{fb}})$
  2. **Arc reversal with LP $(\mathbf{AR}_{\mathbf{lp}})$**
  3. **Variable elimination outer approximation $(\mathbf{VE}_{\mathbf{outer}})$**
  4. **Variable elimination with LP $(\mathbf{VE}_{\mathbf{lp}})$**

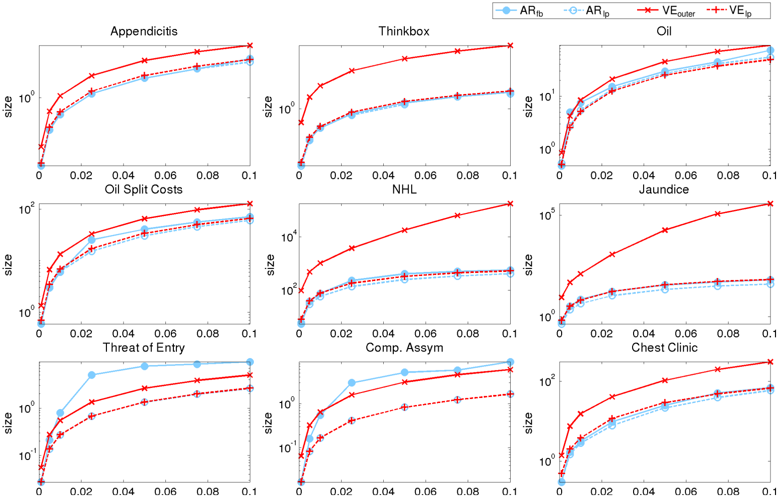
Empirical Validation

- 8 precise IDs were transformed into interval-valued IDs by a perturbation of the original parameters
- The objectives are:
  - Computation time (also comparing with the precise evaluation)
  - A **sensitivity analysis** to evaluate the effect of the size of the intervals of the initial potentials affects the informativeness of the solution.

Empirical Validation - Computation time



Empirical Validation - Size of $EU(\Delta^*)$ for different sizes of the IPPs



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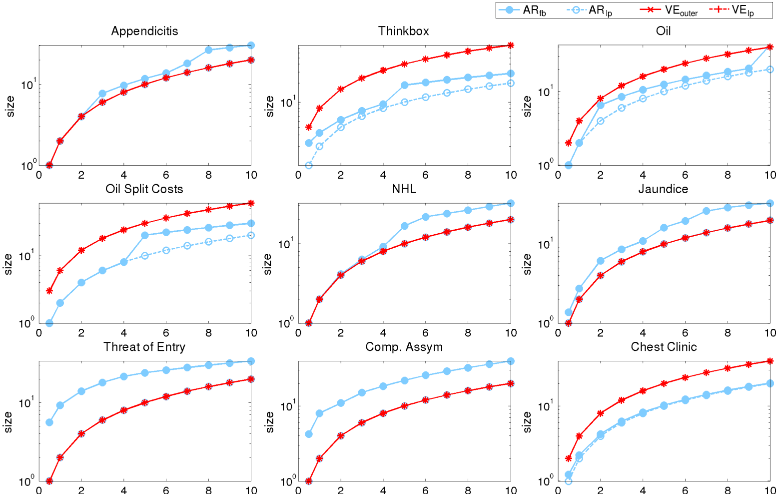

Conclusions and Future Work

- We have generalized the formalism of influence diagrams to the interval framework.
- Algorithms based on VE and AR have been also proposed and compared.
  - The complexity w.r.t. the precise evaluation is not increased.
  - The best results are obtained if the **linear programming (LP)** approach is considered
- **As a future work** we intend to extend IDs to more general imprecise frameworks (credal sets represented by extreme points):
  - Approximate algorithms
  - Cluster computation

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THANK YOU FOR YOUR ATTENTION
GRACIAS POR SU ATENCIÓN

QUESTIONS?

Appendix: optimal policy

Evaluating an ID implies computing:

- The optimal policy $\delta^*_{D_i} : \Omega_{\Pi_{D_i}} \to \Omega_{D_i}$ for each decision D_i:

$$\delta^*_{D_i}(\Pi_{D_i}) = \arg\max_{D_i} \sum_{\mathcal{I}_i} \max_{D_{i+1}} \cdots \max_{D_n} \sum_{\mathcal{I}_n} \left(\prod_{X \in \boldsymbol{X}} \phi(X|\Pi_X) \sum_{U \in \boldsymbol{U}} \psi(\Pi_U) \right)$$

- The expected utility of the optimal strategy $\Delta^* = \{\delta^*_{D_1}, \ldots, \delta^*_{D_n}\}$

$$EU(\Delta^*) = \sum_{\mathcal{I}_0} \max_{D_1} \cdots \max_{D_n} \sum_{\mathcal{I}_n} \prod_{X \in \boldsymbol{X}} \phi(X|\Pi_X) \sum_{U \in \boldsymbol{U}} \psi(\Pi_U)$$

		Seismic		
	δ^*_D	c	o	d
Test	t	d	d	nd
	nt	d	d	d

$EU(\Delta^*) = 22.5$

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$$EU(\Delta^*) = 22.5$$

Appendix: conditions for IPPs

- Non-empty extension:
  $\underline{\phi}(x_I|x_J) \leq \overline{\phi}(x_I|x_J) \;\; \forall (x_I, x_J) \in \Omega_{X_I} \times \Omega_{X_J}$
  $\sum_{x_I} \underline{\phi}(x_I|x_J) \leq 1 \leq \sum_{x_I} \overline{\phi}(x_I|x_J) \;\; \forall x_J \in \Omega_{X_J}$
- An IPP is **reachable** if only if:
  $\overline{\phi}(x_i') + \sum_{x_i \neq x_i'} \underline{\phi}(x_i) \leq 1$
  $\underline{\phi}(x_i') + \sum_{x_i \neq x_i'} \overline{\phi}(x_i) \geq 1$

Any IPP with non-empty extension can be always reduced to a reachable one by shrinking its bounds.